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Mean curvature deficit and a quasi-local mass

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ABSTRACT. We give an account of how theorems of Wang-Yau [16] and Shi-Tam [14] concerning the boundary mean curvature of compact 3-manifolds with a negative lower bound on scalar curvature give rise to a new intrinsic metric invariant on 2-spheres. This invariant was used to derive a functional that extends Brown-York quasi-local mass beyond the setting of positively curved boundaries. We compare this functional to the Hawking quasi-local mass, extending known results about Brown-York mass in [15, 10].

1. Introduction

Let (Ω, g) be a compact Riemannian 3-manifold whose boundary is a 2-sphere (Σ, γ) with positive Gauss curvature. Under the assumptions that Ω has nonnegative scalar curvature and $\partial\Omega$ has positive mean curvature, Shi and Tam [13] proved that the Brown-York quasi-local mass [1, 2]

$$\mathfrak{m}_{BY}(\Sigma; \Omega) = \frac{1}{8\pi} \int_{\partial\Omega} (H_0 - H)\, d\sigma$$

is nonnegative, and vanishes if and only if (Ω, g) is isometric to a domain in $\mathbb{R}^3$. Here $d\sigma$ denotes the induced area element on $\partial\Omega = \Sigma$, H the mean curvature of $\partial\Omega$ in Ω, and H_0 the mean curvature of the isometric embedding of (Σ, γ) into $\mathbb{R}^3$, which exists and is unique (up to rigid motions) by the solution to Weyl's embedding problem [11, 12].

Denote the isometric embedding of (Σ, γ) into $\mathbb{R}^3$ by ι. Cast in a different light, Shi and Tam's theorem asserts that the quantity

$$\int_{\iota(\Sigma)} H_0\, d\sigma,$$

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which is an intrinsic invariant of the 2-sphere (Σ, γ) with positive Gauss curvature, serves as the supremum for the total boundary mean curvature functional,

$$(\Omega, g) \mapsto \int_{\partial\Omega} H \, d\sigma,$$

among a suitable class of "fill-ins" (Ω, g) of (Σ, γ). This characterization led the authors to the following definition:

DEFINITION 1.1 ([**8**]). Given a Riemannian metric γ on a 2-sphere Σ, denote by $\mathcal{F}_{(\Sigma,\gamma)}$ the set of all compact, connected Riemannian 3-manifolds (Ω, g) with boundary such that:

(1) $\partial\Omega$, with its induced metric, is isometric to (Σ, γ),
(2) $\partial\Omega$ is mean-convex; i.e., the mean curvature H of $\partial\Omega$ with respect to the outward normal is positive, and
(3) $R_g \geq 0$, where R_g is the scalar curvature of g.

Moreover, define

$$\Lambda_{(\Sigma,\gamma)} := \sup \left\{ \frac{1}{8\pi} \int_{\partial\Omega} H \, d\sigma \mid (\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma)} \right\}.$$

When the Gauss curvature K_γ of γ is positive, it follows from Shi-Tam's theorem [**13**] that $\Lambda_{(\Sigma,\gamma)}$ is finite and, moreover, equals

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_{\iota(\Sigma)} H_0 \, d\sigma. \tag{1.1}$$

In this case, the Brown-York mass of Σ in (Ω, g) takes an equivalent form of

$$\mathfrak{m}_{BY}(\Sigma; \Omega) = \Lambda_{(\Sigma,\gamma)} - \frac{1}{8\pi} \int_{\Sigma} H \, d\sigma. \tag{1.2}$$

Motivated by (1.2), the authors proposed the following variant of Brown-York mass, which is defined without the restriction on the Gauss curvature of Σ.

DEFINITION 1.2 ([**8**]). Let (Ω, g) be a compact, connected, mean-convex Riemannian 3-manifold with scalar curvature $R_g \geq 0$. Suppose $\Sigma = \partial\Omega$ is a 2-sphere. Define

$$\mathfrak{m}(\Sigma; \Omega) := \Lambda_{(\Sigma,\gamma)} - \frac{1}{8\pi} \int_{\Sigma} H \, d\sigma. \tag{1.3}$$

Here $d\sigma$ is the induced area form on Σ.

By virtue of (1.1) and (1.2), the functional $\mathfrak{m}(\Sigma; \Omega)$ extends Brown-York mass, $\mathfrak{m}_{BY}(\Sigma; \Omega)$, since

$$\mathfrak{m}(\Sigma; \Omega) = \mathfrak{m}_{BY}(\Sigma; \Omega) \text{ if } K_\gamma > 0. \tag{1.4}$$

The following proposition underlies the definition of $\mathfrak{m}(\Sigma; \Omega)$.

PROPOSITION 1.1 ([**8**]). *For every (Ω, g) as in Definition 1.2,*

$$0 \le \mathfrak{m}(\Sigma; \Omega) < \infty.$$

If $\mathfrak{m}(\Sigma; \Omega) = 0$, then Ω is a 3-ball and g is flat.

Another important quasi-local mass functional in the literature is the Hawking quasi-local mass [**5**]. Given a surface Σ in a 3-manifold (Ω, g), the Hawking quasi-local mass $\mathfrak{m}_{H}(\Sigma; \Omega)$ is given by

$$\mathfrak{m}_{H}(\Sigma; \Omega) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\sigma \right).$$

Here $|\Sigma|_\gamma$ denotes the area of (Σ, γ). Note that, as with $\mathfrak{m}(\Sigma; \Omega)$, there are no requirements on the Gauss curvature of Σ in the definition of $\mathfrak{m}_{H}(\Sigma; \Omega)$.

If $K_\gamma > 0$ and the region (Ω, g) that (Σ, γ) bounds is mean-convex with nonnegative scalar curvature, it was shown in [**15**] (also cf. [**10**]) that the Brown-York mass and Hawking mass satisfy

$$\mathfrak{m}_{BY}(\Sigma; \Omega) \ge \mathfrak{m}_{H}(\Sigma; \Omega). \tag{1.5}$$

Having extended $\mathfrak{m}_{BY}(\Sigma; \Omega)$ to $\mathfrak{m}(\Sigma; \Omega)$, one naturally wonders if (1.5) continues to hold with $\mathfrak{m}_{BY}(\Sigma; \Omega)$ replaced by $\mathfrak{m}(\Sigma; \Omega)$. In this note we show that this is indeed the case for a wide class of metrics that contains those with positive Gauss curvature.

THEOREM 1.3. *Suppose (Ω, g) is a compact Riemannian 3-manifold with nonnegative scalar curvature, and with mean-convex boundary Σ that is a 2-sphere. Let γ be the induced metric on Σ. If (Σ, γ) is isometric to an outer-minimizing, mean convex surface in $\mathbb{R}^3$, then*

$$\mathfrak{m}(\Sigma; \Omega) \ge \mathfrak{m}_{H}(\Sigma; \Omega) \tag{1.6}$$

with equality if and only if (Ω, g) is isometric to a geodesic ball in $\mathbb{R}^3$.

REMARK 1.4. Star-shaped, mean convex surfaces in $\mathbb{R}^3$ are automatically outer-minimizing. Hence, Theorem 1.3 applies to all (Ω, g) whose boundaries are isometric to such surfaces, generalizing (1.5).

2. The $\Lambda_{(\Sigma,\gamma)}$ invariant

In [**8**], $\Lambda_{(\Sigma,\gamma)}$ was defined and investigated for all closed, orientable, possibly disconnected surfaces Σ, endowed with a Riemannian metric γ. Here we focus on the case when Σ is a 2-sphere and collect known properties of $\Lambda_{(\Sigma,\gamma)}$ in Proposition 2.1; (i), (ii), (iii) can already be found in [**8**]; (iv) is a new fact and will be proven immediately following Remark 2.2 below. Note that (i), (ii) together imply Proposition 1.1 in the introduction, and that (iii) reiterates (1.1).

PROPOSITION 2.1. *Suppose Σ is a 2-sphere, endowed with a Riemannian metric γ. Then:*

(i) *$\Lambda_{(\Sigma,\gamma)}$ is finite whenever $\mathcal{F}_{(\Sigma,\gamma)} \neq \emptyset$.*
(ii) *If $\Lambda_{(\Sigma,\gamma)}$ is attained by some $(\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma)}$, i.e., if*

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_{\partial\Omega} H \, d\sigma,$$

then Ω is a 3-ball and g is flat.
(iii) *If (Σ, γ) is isometric to a strictly convex surface Σ_0 in $\mathbb{R}^3$ with mean curvature H_0, then*

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_{\iota(\Sigma)} H_0 \, d\sigma.$$

(iv) *The value $\Lambda_{(\Sigma,\gamma)}$ can be estimated from below if (Σ, γ) is isometric to a mean-convex, outer-minimizing 2-sphere in $\mathbb{R}^3$. In this case,*

$$\Lambda_{(\Sigma,\gamma)} \geq \sqrt{\frac{|\Sigma|_\gamma}{4\pi}},$$

where $|\Sigma|_\gamma$ denotes the area of (Σ, γ). Moreover, equality holds if and only if (Σ, γ) is isometric to a round sphere in $\mathbb{R}^3$.

REMARK 2.1. (i) is an a priori L^1 bound on the boundary mean curvature of mean-convex (Ω, g) with spherical boundary and nonnegative scalar curvature, and it depends only on the induced intrinsic boundary geometry. Such an a priori estimate is crucial to the definition and study of $\mathfrak{m}(\Sigma; \Omega)$. Its proof is particularly interesting, and perhaps unconventional in the study of nonnegative scalar curvature, in that it requires the study of asymptotically hyperbolic—not Euclidean—manifolds. Such a study was carried out by Wang-Yau [**16**] and Shi-Tam [**14**] to understand the boundary mean curvature of compact manifolds whose scalar curvature has a negative lower bound. Using that work, the L^1 estimate was derived in [**8**] for Σ consisting of one or more 2-spheres. The same estimate, for a single 2-sphere, was derived independently by Lu [**7**]; in Lu's work it serves as a geometric inequality that allows for a priori estimates of geometric quantities via elliptic PDE techniques. We will survey the foundational results of Wang-Yau [**16**] and Shi-Tam [**14**] that lead to this L^1 bound at the end of this section.

REMARK 2.2. (ii) is a rigidity statement for maximizers of total mean curvature. Let us outline the proof. First, one shows that a maximizer (Ω, g) is scalar-flat by analyzing competitors in its conformal class. Second, one shows that (Ω, g) is Ricci-flat by analyzing critical points of $\int_{\partial\Omega} H d\sigma$ among scalar-flat metrics. In three dimensions, g is then flat. By a general variational argument, one knows that mean-convex and Ricci-flat (Ω, g) contain no interior minimal hypersurfaces. The topological conclusion, therefore, follows from the work of Meeks-Simon-Yau [**9**]. We refer the reader to [**8**] for more details, as well as a more general rigidity result, for Σ consisting of one or more 2-surfaces.

PROOF OF PROPOSITION 2.1 (iv). Suppose (Σ, γ) is isometric to Σ_0, a mean-convex, outer-minimizing surface in $\mathbb{R}^3$. Let $\Omega_0 \subset \mathbb{R}^3$ be the domain bounded by Σ_0. By definition, $(\Omega_0, g_0) \in \mathcal{F}_{(\Sigma,\gamma)}$ where g_0 denotes the Euclidean metric. Therefore,

$$\Lambda_{(\Sigma,\gamma)} \geq \frac{1}{8\pi} \int_{\Sigma_0} H_0 \, d\sigma. \tag{2.1}$$

On the other hand, if H_0 is the mean curvature of Σ_0 and $|\Sigma_0|$ denotes the area of Σ_0,

$$\frac{1}{8\pi} \int_{\Sigma_0} H_0 d\sigma \geq \sqrt{\frac{|\Sigma_0|}{4\pi}} = \sqrt{\frac{|\Sigma|_\gamma}{4\pi}}, \tag{2.2}$$

where the inequality is the Aleksandrov-Fenchel inequality for outer-minimizing surfaces (see Theorem 6 in [4] due to Huisken; also see [**3**]). Moreover, equality in (2.2) holds if and only if Σ_0 is a round sphere. The result now follows from (2.1) and (2.2). □

We now conclude this section by recalling the work of Wang-Yau [**16**] and Shi-Tam [**14**], whose results provide the essential cause for the finiteness of $\Lambda_{(\Sigma,\gamma)}$, i.e., the a priori L^1 bound on H.

Recall from the introduction that, when $K_\gamma > 0$, the total mean curvature H_0 of the isometric embedding of (Σ, γ) into Euclidean space dominates the total boundary mean curvature H of all mean-convex fill-ins (Ω, g) with $R_g \geq 0$. This follows from the work of Shi-Tam [**13**].

Such an argument cannot go through without the $K_\gamma > 0$ assumption, as one may no longer be able to embed (Σ, γ) isometrically in Euclidean space. However, Pogorelov [**12**] showed that (Σ, γ) will always embed isometrically inside hyperbolic space, $\mathbb{H}^3_{-\kappa^2}$, provided we pick a sectional curvature $-\kappa^2 < \inf K_\gamma$. This embedding is unique up to ambient isometries of $\mathbb{H}^3_{-\kappa^2}$.

In studying the boundary mean curvature of compact, mean-convex manifolds (Ω, g) with $R_g \geq -6\kappa^2$ and spherical boundary (Σ, γ) with $\inf_\Sigma K_\gamma > -\kappa^2$, Wang-Yau [**16**] deduced that a particular weighted average of the mean curvature H_0 of the isometric embedding of (Σ, γ) into $\mathbb{H}^3_{-\kappa^2}$ dominates the corresponding weighted average of the mean curvature H of $\partial\Omega$. Specifically, they proved:

THEOREM 2.3 ([**16**]). *Let (Ω, g) be a compact, Riemannian 3-manifold with scalar curvature $R \geq -6\kappa^2$ for some $\kappa > 0$, and boundary Σ. Suppose Σ is a 2-sphere with Gauss curvature $K > -\kappa^2$. Suppose the mean curvature H of Σ with respect to the outward unit normal is positive. Then there exists a future-directed time-like vector-valued function $\mathbf{W}^0 : \Sigma \longrightarrow \mathbb{R}^{3,1}$, which depends on H and the isometric embedding of Σ into $\mathbb{H}^3_{-\kappa^2} \subset \mathbb{R}^{3,1}$, such that*

$$\int_\Sigma (H_0 - H) \mathbf{W}^0 \, d\sigma \tag{2.3}$$

is a future-directed non-spacelike vector. Here H_0 is the mean curvature of Σ when it is isometrically embedded in $\mathbb{H}^3_{-\kappa^2}$ and $d\sigma$ denotes the area element on Σ.

In this theorem, $\mathbb{H}^3_{-\kappa^2}$ is identified with the hyperboloid

$$\{(x_1, x_2, x_3, t) \mid t^2 - x_1^2 - x_2^2 - x_3^2 = \kappa^{-2},\, t > 0\} \subset \mathbb{R}^{3,1},$$

where $\mathbb{R}^{3,1}$ is the 4-dimensional Minkowski spacetime. The vector-valued function $\mathbf{W}^0$ in Theorem 2.3, i.e., the weight, was produced by an elegant construction which involves solving a backward heat equation with the initial value prescribed at infinity.

In [**14**], by modifying the argument in [**16**], Shi and Tam discovered that $\mathbf{W}^0$ can be taken as a variation of the position vector of the embedding of (Σ, γ) in $\mathbb{H}^3_{-\kappa^2}$. Precisely, they proved

THEOREM 2.4 ([**14**]). *Let (Ω, g) be as in Theorem 2.3. Let $X = (x_1, x_2, x_3, t) : \Sigma \to \mathbb{H}^3_{-\kappa^2} \subset \mathbb{R}^{3,1}$ be an isometric embedding of Σ such that its image $X(\Sigma)$ encloses the point $(0, 0, 0, \kappa^{-1}) \in \mathbb{H}^3_{-\kappa^2}$. There exists a constant $\alpha > 1$, depending only on the embedding X, such that*

$$\int_\Sigma (H_0 - H)\mathbf{W}_\Sigma \cdot \zeta \, d\sigma \le 0 \tag{2.4}$$

for any future-directed null vector $\zeta \in \mathbb{R}^{3,1}$, where $\mathbf{W}_\Sigma = (x_1, x_2, x_3, \alpha t)$. Moreover, if equality in (2.4) holds for some ζ, (Ω, g) is isometric to a domain in $\mathbb{H}^3_{-\kappa^2}$.

As a corollary of Theorem 2.4, Shi and Tam obtained

COROLLARY 2.1 ([**14**]). *Let (Ω, g) be as in Theorem 2.3. Suppose D is the bounded region enclosed by Σ when Σ is isometrically embedded in $\mathbb{H}^3_{-\kappa^2}$. Then for any point $o \in D$, letting r be the geodesic distance from o on $\mathbb{H}^3_{-\kappa^2}$, we have*

$$\int_\Sigma (H_0 - H) \cosh \kappa r \, d\sigma \ge 0 \tag{2.5}$$

and equality holds if and only if (Ω, g) is isometric to a domain in $\mathbb{H}^3_{-\kappa^2}$.

Now we are in a position to return to our setting of nonnegative scalar curvature and explain why $\Lambda_{(\Sigma,\gamma)} < \infty$. Given any $(\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma)}$, choose a constant $\kappa > 0$, depending only on (Σ, γ), such that $K > -\kappa^2$. Since the scalar curvature R of (Ω, g) is nonnegative, one has $R \ge -6\kappa^2$. Therefore, Corollary 2.1 applies to (Ω, g) to assert

$$\int_\Sigma H d\sigma \le \int_\Sigma H \cosh \kappa r \, d\sigma \le \int_\Sigma H_0 \cosh \kappa r \, d\sigma,$$

where H_0 is the mean curvature of the isometric embedding X of (Σ, γ) in $\mathbb{H}^3_{-\kappa^2}$ and r is the geodesic distance in $\mathbb{H}^3_{-\kappa^2}$ to a fixed point contained inside the region enclosed by $X(\Sigma)$. By the work of Pogorelov [**12**], the right hand side above is intrinsic to (Σ, γ). Taking the supremum over all $(\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma)}$,

$$\Lambda_{(\Sigma,\gamma)} \le \frac{1}{8\pi} \int_\Sigma H_0 \cosh \kappa r \, d\sigma < \infty.$$

3. Comparison between $\mathfrak{m}(\Sigma;\Omega)$ and $\mathfrak{m}_{_H}(\Sigma;\Omega)$

We now focus on the mass functional, $\mathfrak{m}(\Sigma;\Omega)$, which we intend to compare to the Hawking quasi-local mass, $\mathfrak{m}_{_H}(\Sigma;\Omega)$. The following proposition, which is interesting in its own right, will lead to the proof of Theorem 1.3.

PROPOSITION 3.1. *Suppose (Ω, g) is a compact Riemannian 3-manifold with nonnegative scalar curvature, and with mean-convex boundary Σ that is a 2-sphere. If the induced metric γ on Σ satisfies*

$$\Lambda_{(\Sigma,\gamma)} \ge \sqrt{\frac{|\Sigma|_\gamma}{4\pi}}, \tag{3.1}$$

then

$$\mathfrak{m}(\Sigma;\Omega) \ge \mathfrak{m}_{_H}(\Sigma;\Omega) \tag{3.2}$$

with equality if and only if (Ω, g) is isometric to a geodesic ball in $\mathbb{R}^3$.

PROOF. We have

$$\begin{aligned} \mathfrak{m}(\Sigma;\Omega) &= \Lambda_{(\Sigma,\gamma)} \left(1 - \frac{1}{8\pi\Lambda_{(\Sigma,\gamma)}} \int_\Sigma H\, d\sigma\right) \\ &\ge \frac{\Lambda_{(\Sigma,\gamma)}}{2} \left[1 - \left(\frac{1}{8\pi\Lambda_{(\Sigma,\gamma)}} \int_\Sigma H\, d\sigma\right)^2\right] \\ &\ge \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left[1 - \left(\frac{1}{8\pi\Lambda_{(\Sigma,\gamma)}} \int_\Sigma H\, d\sigma\right)^2\right]. \end{aligned} \tag{3.3}$$

The first inequality above follows from $1 - x \ge \frac{1}{2}(1 - x^2)$, $\forall\, x \in \mathbb{R}$. The second inequality follows from assumption (3.1) and the fact

$$\int_\Sigma H\, d\sigma \le 8\pi\Lambda_{(\Sigma,\gamma)}.$$

From Hölder's inequality and assumption (3.1) on $\Lambda_{(\Sigma,\gamma)}$, one also has

$$\left(\int_\Sigma H\, d\sigma\right)^2 \le |\Sigma|_\gamma \int_\Sigma H^2\, d\sigma \le 4\pi\Lambda^2_{(\Sigma,\gamma)} \int_\Sigma H^2\, d\sigma. \tag{3.4}$$

Inequality (3.2) now follows from (3.3) and (3.4).

If equality holds in (3.2), all inequalities in (3.3) and (3.4) are forced to become equalities. In particular, equality in (3.4) implies H is a constant and equality in (3.3) shows

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_\Sigma H d\sigma, \tag{3.5}$$

i.e., $\mathfrak{m}(\Sigma;\Omega)=0$. Thus, by Proposition 2.1 (ii), (Ω, g) is a flat 3-ball. Let A be the second fundamental form of Σ in (Ω, g). Its traceless part,

$$\mathring{A} := A - \frac{1}{2}H\gamma,$$

is divergence free on Σ by the Codazzi equation and the fact H is a constant. Since Σ is a 2-sphere, any trace-free and divergence-free $(0,2)$ symmetric tensor on Σ must vanish (cf. [**6**]). Hence, $\mathring{A} = 0$, which shows $A = \frac{1}{2}H\gamma$. By the Gauss equation, one has $K_\gamma = \frac{1}{4}H^2$, which is a positive constant. Therefore, (Σ, γ) is isometric to a round sphere in $\mathbb{R}^3$. In particular, $K_\gamma > 0$, so Brown-York mass is well-defined and

$$\mathfrak{m}_{BY}(\Sigma;\Omega) = \mathfrak{m}(\Sigma;\Omega) = 0.$$

It follows that (Ω, g) is isometric to a Euclidean geodesic ball. □

REMARK 3.1. In the above proof, we only used the fact that Σ is a 2-sphere to conclude $\mathring{A}=0$. In general, for genus g-surfaces Σ the proof shows that

$$\Lambda_{(\Sigma,\gamma)} - \frac{1}{8\pi}\int_{\partial\Omega} H\, d\sigma \geq \mathfrak{m}_{H}(\Sigma;\Omega),$$

with equality if and only if both sides of the inequality vanish, H is constant, and equality holds in assumption (3.1).

PROOF OF THEOREM 1.3. The Theorem follows directly from Proposition 2.1 (iv) and Proposition 3.1. □

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