References

[1] David Baraglia, Hokuto Konno, *A gluing formula for families Seiberg-Witten invariants*, Geom. Topol. **24** (2020), 1381–1456.

[2] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. I*, J. Amer. Math. Soc. **24** (2018), 215–264.

[3] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. II*, Ann. of Math. **182** (2017), 127–204.

[4] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. **2** (1985), 215–249.

[5] Hokuto Konno, *Characteristic classes via 4-dimensional gauge theory*, Geom. Topol. **25** (2021), 711–773.

[6] Hokuto Konno, Jianfeng Lin, *Homological instability for moduli spaces of smooth 4-manifolds*, arXiv:2211.03043.

[7] Ib Madsen, Michael Weiss, *The stable moduli space of Riemann sur- faces: Mumford's conjecture*, Ann. of Math. **165** (2007), 843–941.

[8] Daniel Ruberman, *An obstruction to smooth isotopy in dimension* 4, Math. Res. Lett. **5** (1998), 743–758.

[9] Daniel Ruberman, *Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants*, Geom. Topol. **5** (2001), 895–924.

[10] M. Muster, *Computing other invariants of topological spaces of dimension three*, Topology **32** (1990), 120–140.

Improved generic regularity of minimizing hypersurfaces

Christos Mantoulidis

(joint work with Otis Chodosh, Felix Schulze)

Let Γ be a smooth, closed, oriented, $(n-1)$-dimensional submanifold of $\mathbf{R}^{n+1}$. Among all smooth, compact, oriented hypersurfaces $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, does there exist one with *least area*?

Foundational results in geometric measure theory can be used to produce an integral n-current T with least mass ("minimizing") among all those with boundary equal to the multiplicity-one current represented by Γ. When $n+1 \leq 7$, it is known that T is supported on a smooth, compact, oriented hypersurface that solves the original differential geometric problem (see [1, 2, 3, 4, 5]). When $n+1 \geq 8$, smooth minimizers can fail to exist (see [6]) but it is nevertheless known that away from a compact set $\operatorname{sing} T \subset \mathbf{R}^{n+1} \setminus \Gamma$ of Hausdorff dimension $\leq n-7$, the support of T will be a smooth precompact hypersurface with boundary Γ (see [7, 5]).

A fundamental result of Hardt–Simon [8] shows that the singularities of 7-dimensional minimizing currents in $\mathbf{R}^8$, which are necessarily isolated points, can be eliminated by a perturbation of the prescribed boundary Γ, thus yielding solutions to the original geometric problem in $\mathbf{R}^8$ for the perturbed boundary.

In recent work motivated from our past results on mean curvature flow (see, e.g., [9, 10]) we obtained a generic regularity result for minimizers in higher ambient dimensions:

Theorem ([11], [12]). *Let $\Gamma^{n-1} \subset \mathbf{R}^{n+1}$ be a smooth, closed, oriented, submanifold. There exist arbitrarily small perturbations Γ' of Γ such that every minimizing integral n-current with boundary $[\![\Gamma']\!]$ is of the form $[\![M']\!]$ for a smooth, precompact, oriented hypersurface M' with $\partial M' = \Gamma'$ and $\operatorname{sing} M' = \bar{M}' \setminus M'$ satisfies*

$$\operatorname{sing} M' = \emptyset \text{ if } n+1 \leq 10, \text{ otherwise } \dim_H \operatorname{sing} M' \leq n - 9 - \varepsilon_n$$

where $\varepsilon_n \in (0,1]$ is an explicit dimensional constant.

Let us discuss what goes into the proof of this theorem. Let us denote

$$\mathcal{M}(\Gamma) = \{\text{minimizing integral } n\text{-currents in } \mathbf{R}^{n+1} \text{ with boundary } [\![\Gamma]\!]\}.$$

We agree to the following simplifying assumptions (see [12] for the general case):

- Γ is connected.
- $\mathcal{M}(\Gamma)$ is a singleton.

The above and the standard regularity theory guarantee that $\mathcal{M}(\Gamma) = \{[\![M]\!]\}$ for a smooth, precompact, oriented hypersurface $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, $\operatorname{sing} M = \bar{M} \setminus M \subset\subset \mathbf{R}^{n+1} \setminus \Gamma$, and $\dim_H \operatorname{sing} M \leq n - 7$.

Now set $\Gamma_0 := \Gamma$ and perturb Γ smoothly to $(\Gamma_s)_{s \in (-\delta,\delta)}$ by s times the unit normal to M along Γ (recall that $\operatorname{sing} M \cap \Gamma = \emptyset$) for some small $\delta > 0$. Accordingly, for each $s \in (-\delta, \delta)$, let $\mathcal{M}(\Gamma_s)$ be the set of all minimizers with boundary data Γ_s; each such is still of the form $[\![M_s]\!]$, with M_s enjoying similar a priori regularity as M. A cut-and-paste argument implies that

$$(\ddagger) \qquad [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ [\![M_{s'}]\!] \in \mathcal{M}(\Gamma_{s'}),\ s \neq s' \implies \bar{M}_s \cap \bar{M}_{s'} = \emptyset.$$

Define

$$\mathcal{L} = \cup_{s \in (-\delta,\delta)} \cup_{[\![M_s]\!] \in \mathcal{M}(\Gamma_s)} \bar{M}_s,$$
$$\mathcal{S} = \cup_{s \in (-\delta,\delta)} \cup_{[\![M_s]\!] \in \mathcal{M}(\Gamma_s)} \operatorname{sing} M_s.$$

In view of ($\ddagger$), the following "timestamp" function is well-defined:

$$\mathbf{t} : \mathcal{L} \to (-\delta, \delta),$$
$$\mathbf{t}(x) = s \text{ for all } x \in \bar{M}_s,\ [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ s \in (-\delta, \delta).$$

We are now ready to state the two main tools required for our main theorem.

Tool A ([12]). *It holds that $\dim_H \mathcal{S} \leq n - 7$.*

Tool B ([12]). *The timestamp function $\mathbf{t} : \mathcal{L} \to (-\delta, \delta)$ above is α-Hölder on $\mathcal{S}$ for every $\alpha \in (0, 2 + \varepsilon_n)$, where $\varepsilon_n \in (0,1]$ is an explicit dimensional constant.*

To obtain the **Theorem** from **Tools A, B** one can invoke a Sard-type covering argument of Figalli–Ros-Oton–Serra, who successfully proved a generic regularity result for free boundary singularities in the obstacle problem using tools similar to **A, B**.

Proposition ([13, Proposition 7.7]). *Let $S \subset \mathbf{R}^n$, $0 < d \leq n$, and $0 < \beta < \alpha$. Assume that $\mathcal{H}^d(S) < \infty$ and that $f : S \to (-1, 1)$ is α-Hölder continuous.*

(1) If $d \leq \beta$, then $\mathcal{H}^{d/\beta}(f(S)) = 0$.
(2) If $d > \beta$, then for a.e. $t \in (-1, 1)$ we have $\mathcal{H}^{d-\beta}(f^{-1}(t)) = 0$.

References

[1] W. H. Fleming. *On the oriented Plateau problem.* Rend. Circ. Mat. Palermo (2) **11** (1962), 69–90.

[2] E. De Giorgi. *Una estensione del teorema di Bernstein.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **19** (1965), 79–85.

[3] F. J. Almgren, Jr. *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem.* Ann. of Math. (2) **84** (1966), 277–292.

[4] J. Simons. *Minimal varieties in riemannian manifolds.* Ann. of Math. (2) **88** (1968), 62–105.

[5] R. Hardt and L. Simon. *Boundary regularity and embedded solutions for the oriented Plateau problem.* Ann. of Math. (2) **110** (1979), 439–486.

[6] E. Bombieri, E. De Giorgi, and E. Giusti. *Minimal cones and the Bernstein problem.* Invent. Math. **7** (1969), 243–268.

[7] H. Federer. *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension.* Bull. Amer. Math. Soc. **76** (1970), 767–771.

[8] R. Hardt and L. Simon. *Area minimizing hypersurfaces with isolated singularities.* J. Reine Angew. Math. **362** (1985), 102–129.

[9] O. Chodosh, K. Choi, C. Mantoulidis, and F. Schulze. *Mean curvature flow with generic initial data I.* arXiv:2003.14344 (2020).

[10] O. Chodosh, K. Choi, C. Mantoulidis, and F. Schulze. *Mean curvature flow with generic low-entropy initial data.* arXiv:2102.11978 (2021).

[11] O. Chodosh, C. Mantoulidis, and F. Schulze. *Generic regularity for minimizing hypersurfaces in dimensions 9 and 10.* arXiv:2302.02253 (2023).

[12] O. Chodosh, C. Mantoulidis, and F. Schulze. *Improved generic regularity for minimizing integral n-currents.* arXiv:2306.13191 (2023).

[13] A. Figalli, X. Ros-Oton, and J. Serra. *Generic regularity of free boundaries for the obstacle problem.* Publ. Math. Inst. Hautes Études Sci. **132** (2020), 181–292.

Diameter estimates in Kähler geometry

Bin Guo

(joint work with Duong H. Phong, Jian Song, and Jacob Sturm)

The diameter is an important geometric invariant associated to a Riemannian metric. Bounds for the diameter are essential for the study of geometric convergence of a family of Riemannian manifolds. Previously known results require the conditions on the curvature. For example, the Myers' theorem states that the diameter is bounded if the Ricci curvature is bounded below by a *positive* constant. Recently, in [1] we develop a general theory of diameter estimates for Kähler metrics, which in particular does not require any assumptions on the Ricci curvature. The nonlinear analysis of complex Monge-Ampère equations allows us to derive uniform estimates of the Green's functions [2], from which the diameter estimates follow.

Let (X, ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X. Let γ be a non-negative continuous function and $A, B, K > 0$, $p > n$ be given parameters. We define a subset of $\mathcal{W} := \mathcal{W}(X, \omega_X, n, A, p, K, \gamma)$ of the space of Kähler metrics on X by

$$\mathcal{W} = \left\{ \omega : \ (V_\omega)^{-1} \frac{\omega^n}{\omega_X^n} \geq \gamma, \ [\omega] \cdot [\omega_X]^{n-1} \leq A, \ \mathcal{N}_{X,\omega_X,p}(\omega) \leq K \right\},$$