We use Seymour's classification [8] of regular matroids to compute sharp upper bounds for $g^*(M,\lambda)$ for all weighted regular matroids of rank less than or equal to nine which only depend on the rank of M.

For ranks less than or equal to six these bounds easily follow from known results [1]. However for ranks between seven and nine our bounds seem to be new. The important estimates which make the proof of Theorem 3 possible are $g^*(M,\lambda) \leq \frac{1}{3}$ for weighted, regular matroids (M,λ) of rank six and $g^*(M,\lambda) \leq \frac{1}{4}$ for weighted, regular matroids (M,λ) of rank nine.

References

[1] B. Bolobás and E. Szemerédi, *Girth of sparse graphs*, J. Graph Theory **39** (2002), 194–200.

[2] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. of Math. (2) **100** (1974), 307–321.

[3] O. Goertsches and M. Wiemeler, *Positively curved GKM-manifolds* Int. Math. Res. Not. IMRN **2015** (2015), 12015–12041.

[4] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Algebra **91** (1994), 137–142.

[5] L. Kennard, *On the Hopf conjecture with symmetry*, Geom. Topol. **17** (2013), 563–593.

[6] L. Kennard, M. Wiemeler, and B. Wilking *Splitting of torus representations and applications in the Grove symmetry program*, arXiv-Preprint, arXiv:2106.14723 (2021).

[7] J. Nienhaus, *PhD-Thesis*, Universität Münster (2022).

[8] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), 305–359.

[9] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math. **191** (2003), 259–297.

Decomposing 4-manifolds with positive scalar curvature

Christos Mantoulidis

(joint work with Richard H. Bamler, Chao Li)

Recall the following well-known theorem of Schoen–Yau ([8]) and Gromov–Lawson ([3]) that exhibits the richness of the class of manifolds that can carry Riemannian metrics with positive scalar curvature. For convenience and brevity, we will adopt the convention of writing "PSC" for "positive scalar curvature," and will call a manifold "topologically PSC" if it can carry PSC Riemannian metrics.

Theorem. *Let M be a topologically PSC n-manifold with $n \geq 3$. Any manifold obtained from M by performing a sequence of* 0-, 1-, ..., *and/or* $(n-3)$*-surgeries is also topologically PSC.*

The theorem above naturally leads one to ask:

Question. *Can all topologically PSC n-manifolds be built out of "simple" topologically PSC n-manifolds by performing codimension ≥ 3 surgeries?*

Our understanding of 3-manifold topology yields a conclusive answer when $n = 3$. The following theorem follows from the combined work of Schoen–Yau ([8, 9]), Gromov–Lawson ([2]), and Perelman ([5, 6, 7]):

Theorem. *Every closed, oriented, topologically PSC* 3*-manifold can be obtained by performing* 0*-surgeries on a disjoint union of spherical space forms (i.e.,* $\mathbf{S}^3/\Gamma$*'s, where the* Γ*'s are finite subgroups of* $SO(4)$ *acting freely on* $\mathbf{S}^3$*).*

The following new result was presented, obtained by the speaker, R. H. Bamler, and C. Li:

Main Theorem ([1]). *Every closed, oriented, topologically PSC* 4*-manifold* M *can be obtained from a possibly disconnected, closed, oriented, topologically PSC* 4*-orbifold* M' *with isolated singularities such that* $b_1(M') = 0$ *and* $b_2(M') \leq b_2(M)$ *by performing* 0*- and* 1*-surgeries. All* 1*-surgeries are standard manifold ones, but* 0*-surgeries may occur at orbifold points.*

Recall that the jth Betti number $b_j(M')$ of the orbifold M' is defined to be the jth Betti number of M' viewed as a topological space; in our case, this is equivalent to the jth Betti number of the regular part $M'_{\text{reg}} \subset M'$. In the connected case, $b_1(M')$ is the same as the rank of the abelianization of the orbifold fundamental group $\pi_1^{\text{orb}}(M')$. Finally, a 0-surgery occurring at two orbifold points both modeled on $\mathbf{R}^4/\Gamma$ means that the corresponding connected sum operation is performed with a $\mathbf{S}^3/\Gamma$ neck.

Our proof of the Main Theorem relies on the flexibility of two-sided stable minimal hypersurfaces in PSC 4-manifolds manifolds due to the speaker and C. Li ([4]). Specifically, the following metric preparation lemma was necessary:

Metric Preparation Lemma. *Let* Σ *be a two-sided, closed, embedded, stable, minimal hypersurface inside an oriented PSC* 4*-manifold* (M, g)*. Then:*

- (a) Σ *must be topologically PSC and thus obtained by performing* 0*-surgeries on a disjoint union of spherical space forms.*
- (b) *Given any auxiliary PSC metric* σ *on* Σ*, there exists a new PSC metric* $\tilde{g}$ *on* M*, which:*
  - *is isometric to a product cylinder* $(\Sigma, \sigma) \times (-2, 2)$ *in the distance-*2 *tubular neighborhood of* Σ*, and*
  - *coincides with* g *outside a larger tubular neighborhood of* Σ*.*

Let now us outline the proof of the Main Theorem. Endow M with an arbitrary PSC metric. We "exhaust" the codimension-1 homology of M with a two-sided, stable minimal hypersurface Σ. By a now-standard argument of Schoen–Yau, the metric induced on Σ is conformal to a PSC metric. Thus, Σ is topologically the result of 0-surgeries on spherical space forms. With the help of the aforementioned flexibility theory, we locally modify the metric on M to another PSC metric that is locally a product near Σ and induces a "model" PSC metric on Σ. If Σ is merely the disjoint union of spherical space forms $\mathbf{S}^3/\Gamma$, with no 0-surgeries, then our model metrics are all round and simple 3-surgeries on M along the components of Σ yield a 4-orbifold whose b_2 is unchanged and b_1 is trivial (assuming Σ suitably "exhausted" the codimension-1 homology of M). If Σ does involve 0-surgeries, we first undo these using 2-surgeries on M near Σ's 0-surgery neck regions; this may decrease b_2. We have only performed 3- and 2-surgeries on M to get to the orbifold, so M can be obtained from the orbifold via 0- and 1-surgeries, respectively.

We conclude our introduction by posing the following:

Question. *Let M be a closed, oriented, topologically PSC 4-manifold. Can one obtain M from a closed, oriented, topologically PSC 4-orbifold M' with isolated singularities and the property that each component has finite orbifold fundamental group $\pi_1^{orb}(M')$ by performing 0- and 1-surgeries?*

References

[1] R. H. Bamler, C. Li and C. Mantoulidis, *Decomposing 4-manifolds with positive scalar curvature*, arXiv Mathematics e-prints (2022), math/2206.09335.

[2] M. Gromov and H. B. Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196 (1984). MR 720933

[3] M. Gromov and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), no. 3, 423–434. MR 577131

[4] C. Li and C. Mantoulidis, *Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the riemannian penrose inequality*, arXiv Mathematics e-prints (2021), math/2106.15709.

[5] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv Mathematics e-prints (2002), math/0211159.

[6] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv Mathematics e-prints (2003), math/0303109.

[7] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv Mathematics e-prints (2003), math/0307245.

[8] R. Schoen, S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183. MR 535700

[9] R. Schoen, S.-T. Yau, *The structure of manifolds with positive scalar curvature*, Directions in partial differential equations (Madison, WI, 1985), Publ. Math. Res. Center Univ. Wisconsin, vol. 54, Academic Press, Boston, MA, 1987, pp. 235–242. MR 1013841

Fundamental Gap Estimate for Convex Domains

Guofang Wei

The fundamental (or mass) gap refers to the difference between the first two eigenvalues of the Laplacian or more generally for Schrödinger operators. It is a very interesting quantity both in mathematics and physics as the eigenvalues are possible allowed energy values in quantum physics. Naturally one looks for optimal upper and lower estimates for the gap. For convex domains with Neumann boundary condition, this is well studied and optimal lower bound has been obtained awhile back. Here we concentrate on the Dirichlet boundary condition.

In their celebrated work, B. Andrews and J. Clutterbuck [1] proved the fundamental gap conjecture that difference of first two eigenvalues of the Laplacian with Dirichlet boundary condition on convex domain with diameter D in the Euclidean space is greater than or equal to $3\pi^2/D^2$. In several joint works with X. Dai, Z. He, S. Seto, L. Wang (in various subsets) [7, 4, 3] the estimate is generalized, showing the same lower bound holds for convex domains in the unit sphere. The key is to prove super log-concavity of the first eigenfunction.