Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the Riemannian Penrose Inequality

Christos Mantoulidis
(joint work with Chao Li)

In all that follows, M denotes a closed n-dimensional manifold and $\mathrm{Met}(M)$ denotes the space of smooth Riemannian metrics on M. For $k \in (0, \infty)$, we define

$$\mathcal{M}_k^{\geq 0}(M) := \{g \in \mathrm{Met}(M) : \lambda_1(-\Delta_g + kR_g) \geq 0\},$$

where $\lambda_1(-\Delta_g + kR_g)$ is the first eigenvalue of the operator $-\Delta_g + kR_g$ on M, and R_g is the scalar curvature of g. We also define

$$\mathcal{M}_\infty^{\geq 0}(M) := \{g \in \mathrm{Met}(M) : R_g \geq 0\}.$$

Finally, we define $\mathcal{M}_k^{>0}(M)$, $k \in (0, \infty]$, as above with all "$\geq$" replaced by "$>$." Note that, for $0 < k < k' \leq \infty$,

$$\begin{array}{ccc} \mathcal{M}_{k'}^{>0}(M) & \subset & \mathcal{M}_{k'}^{\geq 0}(M) \\ \cap & & \cap \\ \mathcal{M}_k^{>0}(M) & \subset & \mathcal{M}_k^{\geq 0}(M). \end{array}$$

These spaces are not generally encountered in the literature in this level of generality, so some remarks are in order about their actual geometric significance. This is discussed extensively in our paper [10]. For the purpose of this brief report, we simply highlight that for $k = \frac{1}{2}$ these spaces encode apparent horizons in time-symmetric initial data sets to Einstein's equations with the dominant energy condition, and that $\mathcal{M}_k^{>0}(M) \neq \emptyset$ for $k = \frac{n-1}{4(n-2)}$ if $n \geq 3$ and M is topologically PSC (i.e., its Yamabe constant is positive).

Our starting point was a generalization of a theorem of Codá Marques [12], who proved that the ultimate space in the filtration has a connected *moduli space*, i.e.,

$$\mathcal{M}_\infty^{>0}(M) / \mathrm{Diff}_+(M) \text{ is path-connected},$$

when M is a closed orientable 3-manifold. We proved:

Theorem 1. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}_k^{>0}(M)/\mathrm{Diff}_+(M)$, $\mathcal{M}_k^{\geq 0}(M)/\mathrm{Diff}_+(M)$ are path-connected for all $k \in [\frac{1}{4}, \infty]$.*

To prove Theorem 1 we needed a suitable generalization of the Gromov–Lawson surgery process [8] (cf. Schoen–Yau's [13]) from $\mathcal{M}_\infty^{>0}(M)$ to $\mathcal{M}_k^{>0}(M)$. Such a surgery was first carried out by Bär–Dahl in [3, Theorem 3.1], and we give a full independent proof of it with some added details in an appendix to our paper.

The recent breakthrough of Bamler–Kleiner [4] on the path-connectedness of $\mathcal{M}_\infty^{>0}(M)$ implies the following two companion results when used in conjunction with Theorem 1 and, separately, the conformal method:

Theorem 2. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}_k^{>0}(M)$ and $\mathcal{M}_k^{\geq 0}(M)$ are path-connected for all $k \in [\frac{1}{4}, \infty]$.*

Theorem 3. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}^{>0}_{1/8}(M)$ is contractible and $\mathcal{M}^{\geq 0}_{1/8}(M)$ is weakly contractible.*

Our main application of these results is to the computation of the Bartnik mass of apparent horizons, and its generalization due to Bray. For n-dimensional closed orientable (M^n, g), the apparent horizon Bartnik mass is defined as

$$\mathfrak{m}_B(M, g, H = 0) = \inf\{\mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g}) : (\mathbf{M}, \mathbf{g}) \in \mathcal{E}_B(M, g, H = 0)\},$$

where $\mathcal{E}_B(M, g, H = 0)$ is the set of complete, connected, asymptotically flat $(\mathbf{M}, \mathbf{g})$ with nonnegative scalar curvature, no closed interior minimal hypersurfaces, and minimal ($H = 0$) boundary isometric to (M, g). Such $(\mathbf{M}, \mathbf{g})$ are initial data sets for solutions of Einstein's equations with the dominant energy condition, and $\mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g})$ is the ADM mass of the initial data set [2, 1]. Using a rearrangement trick of Schoen–Yau and a delicate splitting theorem of Galloway, it follows that:

$$\mathcal{E}_B(M, g, H = 0) \neq \emptyset \implies M \text{ is topologically PSC, } g \in \mathcal{M}^{\geq 0}_{1/2}(M).$$

Thus, we are precisely in the context studied by Theorems 1, 2, 3.

There exists a nontrivial lower bound for $\mathfrak{m}_B(M, g, H = 0)$ by Bray [6] and Bray–Lee's [5] Riemannian Penrose Inequality, which says:

$$(\mathbf{M}, \mathbf{g}) \in \mathcal{E}_B(M, g, H = 0) \implies \mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g}) \geq \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n},$$

when $2 \leq n \leq 6$ and σ_n is the volume of the standard round $\mathbf{S}^n$; see also Huisken–Ilmanen [9] in case $n = 2$ and M is connected. Thus of course

$$\mathfrak{m}_B(M, g, H = 0) \geq \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n},$$

We computed the left hand side to be a topological invariant when $n = 3$ and M is connected. (When $n = 2$, this is due to M.–Schoen [11], Chau–Martens [7].)

Theorem 5. *For a closed connected topologically PSC 3-manifold M, either:*

- $\mathcal{E}_B(M, g, H = 0) = \emptyset$
- $\mathcal{E}_B(M, g, H = 0) \neq \emptyset$ *and* $\mathfrak{m}_B(M, g, H = 0) = \mathfrak{c}_B(M) \operatorname{vol}_g(M)^{2/3}$,

for all $g \in \mathcal{M}^{\geq 0}_{1/2}(M)$. Here, $\mathfrak{c}_B(M)$ is a topological constant and $\mathfrak{c}_B(\mathbf{S}^3) = \frac{1}{2}\sigma_3^{-2/3}$.

Unfortunately, the precise value of the apparent horizon Bartnik mass remains unknown for:

- disconnected 2- or 3-dimensional M;
- 3-dimensional M with nontrivial topology;
- all higher dimensional M, except for certain special metrics on $M = SS^n$.

While we do not have satisfactory answers for the Bartnik mass for these bullet points at this time, we know how to compute a relaxation of Bartnik's mass due to Bray [6] in near-complete generality. In this relaxation, the set $\mathcal{E}_{BB}(M, g, H = 0)$ of extensions considered is such that the boundary (M, g) is outer-minimizing minimal, rather than outermost minimal. The Bartnik–Bray mass $\mathfrak{m}_{BB}(M, g, H = 0)$ is then defined analogously. We showed:

Theorem 6. *Let M be a closed orientable topologically PSC n-manifold with $2 \leq n \leq 6$. Consider the subset of $\mathcal{M}^{\geq 0}_{1/2}(M)$ given by:*

$$\begin{aligned}\mathrm{LinClos}[\mathcal{M}^{>0}_{1/2}(M)] := \{g \in \mathcal{M}^{\geq 0}_{1/2}(M) :\ & \textit{there exists a } C^1 \textit{ path} \\ & [0,1) \ni t \mapsto g(t) \textit{ with } g(0) = g \textit{ and} \\ & \left[\tfrac{d}{dt}\lambda_1(-\Delta_{g(t)} + \tfrac{1}{2}R_{g(t)})\right]_{t=0} > 0\}.\end{aligned}$$

If $g \in \mathrm{LinClos}[\mathcal{M}^{>0}_{1/2}(M)]$ and $\mathcal{E}_{BB}(M, g, H = 0) \neq \emptyset$, then

$$\mathfrak{m}_{BB}(M, g, H = 0) = \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n}.$$

We emphasize that M need not be connected and that our computation is valid as long as a single Bartnik–Bray extension exists. Note that it is known that

$$\mathcal{M}^{\geq 0}_{1/2}(M) \setminus \mathrm{LinClos}[\mathcal{M}^{>0}_{1/2}(M)] \subset \{g \in \mathrm{Met}(M) : \mathrm{Ric}_g \equiv 0\},$$

which is empty when $n = 2$, 3, and small for larger n.

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