The p-widths of a surface

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(joint work with Otis Chodosh)

Fix a closed Riemannian manifold (M^{n+1}, g). The *p-widths* of (M, g), denoted $\omega_p(M, g) \in (0, \infty)$ for $p \in \mathbb{N}^*$, are a geometric nonlinear analogue of the spectrum of its Laplace–Beltrami operator. They are obtained by replacing the Rayleigh quotient of the Laplace–Beltrami operator along families of scalar-valued functions on M with the n-dimensional area along sweepouts of M of (possibly singular) hypersurfaces. They were introduced by Gromov [Gro88, Gro03, Gro09], studied further by Guth [Gut09], and have played a central and exciting role in minimal surface theory when combined with the Almgren–Pitts–Marques–Neves Morse theory program for the area functional. We invite the reader to refer to [Gro88] for the analogy between the Laplace spectrum and the volume spectrum, and to [MN21] for a thorough overview of the importance of this analogy in minimal surface theory.

Let us recall the main existence theorem for p-widths. By the combined work of Almgren–Pitts, Schoen–Simon, Marques–Neves, and Li, it is known that in ambient dimensions $n + 1 \geq 3$ every p-width is attained as the area of a smoothly embedded minimal hypersurface Σ_p whose singular set $\bar{\Sigma}_p \setminus \Sigma_p$ has dimension $\leq n - 7$, whose connected components may contribute to area with different multiplicities, and whose total Morse index (discounting multiplicities) is bounded by p. That is:

Theorem 1 ([Pit81, SS81, MN16, Li20])**.** *Let (M^{n+1}, g) be a closed Riemannian manifold with $n + 1 \geq 3$. For every $p \in \mathbb{N}^*$, there exists a smoothly embedded minimal hypersurface $\Sigma_p \subset M$, with $\bar{\Sigma}_p \setminus \Sigma_p$ of Hausdorff dimension $\leq n - 7$ and components $\Sigma_{p,1}, \ldots, \Sigma_{p,N(p)} \subset \Sigma_p$, such that*

$$\omega_p(M, g) = \sum_{j=1}^{N(p)} m_j \cdot \mathrm{area}_g(\Sigma_{p,j}),$$

where $m_j \in \mathbb{N}^*$ for all $j \in \{1, \ldots, N(p)\}$ and $\mathrm{ind}(\Sigma_p) \leq p$.

Note that, when $3 \leq n + 1 \leq 7$, Σ_p is necessarily smoothly embedded. On the other hand, in the case of a two-dimensional Riemannian manifold ($n + 1 = 2$), min-max methods not only need not produce *embedded* geodesics (see [Aie19] for examples of immersed geodesics being produced), but in full generality they could *a priori* produce *geodesic nets* as opposed to (immersed) geodesics (see [MN16, Remark 1.1]).

Our first main result shows that the min-max methods described above can be guaranteed to produce (immersed) geodesics regardless of the number of parameters. Throughout the paper, a geodesic is said to be primitive if it is connected and traversed with multiplicity one.

Theorem 2. *Let (M^2, g) be a closed Riemannian manifold. For every $p \in \mathbb{N}^*$, there exists a $\sigma_p \subset M$ consisting of primitive closed geodesics $\sigma_{p,1}, \ldots, \sigma_{p,N(p)} \subset \sigma_p$ such that*

$$\omega_p(M, g) = \sum_{j=1}^{N(p)} m_j \cdot \mathrm{length}_g(\sigma_{p,j}),$$

where $m_j \in \mathbb{N}^*$ for all $j \in \{1, \ldots, N(p)\}$.

The existence of immersed geodesics representing the p-widths was previously known for $p = 1$ by Calabi–Cao [CC92] and for $p \in \{1, \ldots, 8\}$ and nearly round metrics on $\mathbf{S}^2$ by Aiex [Aie19].

Our second main result is a computation of the full p-width spectrum of the round two-sphere. (To this point there had not been a single (M^n, g), $n \geq 2$, for which the areas $\omega_p(M, g)$ (let alone the surfaces Σ_p) are known for all $p \in \mathbb{N}^*$, not even in the two-dimensional case. For comparison, the spectrum of the Laplacian is completely determined for a large class of Riemannian manifolds.)

Theorem 3. *Let g_0 denote the unit round metric on $\mathbf{S}^2$. For every $p \in \mathbb{N}^*$,*

$$\omega_p(\mathbf{S}^2, g_0) = 2\pi \lfloor \sqrt{p} \rfloor,$$

and is attained by a sweepout constructed out of homogeneous polynomials. The corresponding σ_p is a union of $\lfloor \sqrt{p} \rfloor$ great circles (repetitions allowed).

One application of Theorem 3 concerns Weyl law for the p-widths. Recall that the Laplacian spectrum (denoted by $\lambda_p(M, g)$) of a closed Riemannian $(n+1)$-manifold satisfies the celebrated Weyl law

$$\lim_{p \to \infty} \lambda_p(M, g) p^{-\frac{2}{n+1}} = 4\pi^2 \,\mathrm{vol}(B)^{-\frac{2}{n+1}} \,\mathrm{vol}(M, g)^{-\frac{2}{n+1}}$$

showing that the high-frequency behavior of the spectrum is universal in a certain sense. Liokumovich–Marques–Neves have recently proven [LMN18] that the p-widths satisfy the following Weyl-type law

$$\lim_{p \to \infty} \omega_p(M, g) p^{-\frac{1}{n+1}} = a(n) \,\mathrm{vol}(M, g)^{\frac{n}{n+1}} \tag{1}$$

for some constant $a(n) > 0$. This result has had important implications for existence of minimal hypersurfaces, cf. [IMN18]. However, the constant $a(n)$ has not been determined for any dimension n (see [LMN18, §1.5]). This is in contrast with the classical Weyl law, where one can use e.g. the (explicitly known) spectrum of a cube to compute the constant in a straightforward manner. Our full computation of the p-widths of the round two-sphere in Theorem 3 readily implies:

Corollary 4. *When $n = 1$, the constant in (1) satisfies $a(1) = \sqrt{\pi}$.*

This settles the "simplest case" of the first question in [LMN18, §1.5].

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Regularity of anisotropic minimal surfaces

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(joint work with Riccardo Tione)

A celebrated theorem of W. Allard [1] states that, given a rectifiable m-varifold V in $\mathbb{R}^N$ with density greater or equal than 1 and generalized mean curvature bounded in $L^p(\|V\|)$ with $p > m$, then V is regular around $x \in \mathbb{R}^N$ provided x has density ratio sufficiently close to 1. The proof deeply relies on the monotonicity formula of the density ratio, which is strictly related to the special symmetries of the area functional, [2]. Hence, it is a hard and widely open question whether this result holds for anisotropic energies, [7, Question 1], i.e. assuming an L^p bound on the anisotropic mean curvature with respect to functionals of the form

$$\Sigma_\Psi(V) := \int_\Gamma \Psi(T_z\Gamma)\theta(z)d\mathcal{H}^m(z), \quad \text{where } V = (\Gamma, \theta) \text{ is a rectifiable } m\text{-varifold.}$$

To the best of our knowledge, the only available result is the regularity for codimension one varifolds with bounded generalized Ψ-mean curvature [3], under a *density lower bound assumption.*