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Ancient gradient flows of elliptic functionals and Morse index

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(joint work with Kyeongsu Choi)

The mean curvature flow is a one-parameter family of submanifolds Σ_t of a Riemannian manifold $(M, \overline{g})$ satisfying the evolution equation

$$\tfrac{\partial}{\partial t}x = \mathbf{H}(x,t),\ x \in \Sigma_t,$$

where $\mathbf{H}(x,t)$ denotes the mean curvature vector of the Σ_t at x, and which is the negative gradient of the area element of Σ_t. As a gradient flow of the area functional, the mean curvature flow describes a natural area minimizing process. In our work [10], we studied closed ancient solutions of the mean curvature flow in Riemannian manifolds; that is, flows of closed submanifolds Σ_t that exist for all $t \in (-\infty, T)$. (We also treated the general case of ancient gradient flows of elliptic functionals, but our strongest and most interesting geometric conclusions were for mean curvature flow.)

Ancient solutions of a gradient flow with uniformly bounded energy are quite rare, and their classification is generally studied as a type of parabolic Liouville theory. There have been a number of important classification results for ancient mean curvature flows inside Euclidean space under assumptions on the convexity or the entropy of the flow. See: X.-J. Wang [16], Huisken–Sinestrari [12], Daskalopoulos–Hamilton–Sesum [11], Angenent–Daskalopoulos–Sesum [2, 3], Brendle–Choi [5, 6], Choi–Haslhofer–Hershkovits [9]. Much less is known in the Riemannian setting. See: Bryan–Louie [8], Bryan–Ivaki–Scheuer [7], Huisken–Sinestrari [12].

In [10] we set up a framework for the characterization of certain types of ancient mean curvature flows in Riemannian manifolds as arising from the "unstable manifold" (i.e., the space of unstable directions for the area functional) of a given closed minimal submanifold. While the framework alone is of independent interest, we also applied it to classify ancient solutions in $\mathbf{S}^n$ under certain natural *area* assumptions:

Theorem 1. *There exists a $\delta = \delta(n) > 0$ such that if $(\Sigma_t)_{t\leq 0}$ is an ancient mean curvature flow of closed m-dimensional surfaces in a round n-sphere $\mathbf{S}^n$, with*

$$\lim_{t\to-\infty} \mathrm{Area}(\Sigma_t) < (1+\delta)\,\mathrm{Area}(\mathbf{S}^m),$$

then $(\Sigma_t)_{t\leq 0}$ is a steady or a canonically shrinking equatorial $\mathbf{S}^m$ along one of $n-m$ directions parallel to the equator.

We emphasize that our result holds true in *arbitrary codimension*, while the previously mentioned results were for codimension-1.

As a corollary to this theorem, we obtained a full classification of ancient embedded flows of curves with uniformly bounded length in $\mathbf{S}^2$ (and, similarly, that there are no nonsteady ancient embedded curve shortening flows with bounded length in flat tori or closed hyperbolic surfaces):

Corollary 1. *Let $(\Gamma_t)_{t\leq 0}$ be an ancient curve shortening flow of embedded closed curves with uniformly bounded length inside a round $\mathbf{S}^2$. Then $(\Gamma_t)_{t\leq 0}$ is a steady or a shrinking equator along circles of latitude.*

We also obtained a stronger classification in the 3-sphere, for which we need to recall the Clifford torus

$$\{(x,y,z,w) \in \mathbf{R}^2 \times \mathbf{R}^2 : x^2+y^2 = z^2+w^2 = \tfrac{1}{2}\} \subset \mathbf{S}^3.$$

This is a smoothly embedded minimal submanifold of $\mathbf{S}^3$ with area $2\pi^2$. By the resolution of the Willmore conjecture by Marques–Neves [13], this is the second smallest area among smooth minimal surfaces, following equatorial $\mathbf{S}^2$ that have area 4π. We showed:

Corollary 2. *Let $(\Sigma_t)_{t\leq 0}$ be an ancient mean curvature flow of closed surfaces in a round 3-sphere, with*

$$A_{-\infty} := \lim_{t\to-\infty} \mathrm{Area}(\Sigma_t) < (1+\delta)2\pi^2.$$

If $\delta > 0$ is sufficiently small, then either $A_{-\infty} = 4\pi$ and $(\Sigma_t)_{t\leq 0}$ is a steady or shrinking equator along spheres of latitude, or $A_{-\infty} = 2\pi^2$ and $(\Sigma_t)_{t\leq 0}$ is a steady or shrinking Clifford torus along a canonical 5-parameter family of ancient flows.

Our classification made use of the following new "canonical family existence" and "strong uniqueness" theorems, which we proved for general ancient gradient flows of elliptic functionals on closed Riemannian manifolds. (We recall that the Morse index of a minimal submanifold is the number of negative eigenvalues of its second variation operator. The Morse index of an equatorial $\mathbf{S}^m$ in $\mathbf{S}^n$ is $n-m$ and of a Clifford torus in $\mathbf{S}^3$ is 5.)

Theorem 2 (Canonical family existence)**.** *Let S be a closed, smoothly embedded minimal submanifold in a Riemannian manifold $(M,\overline{g})$. Let $I \in \mathbf{N}$ denote its*

Morse index. Then there exists an I-parameter family of ancient mean curvature flows on $(-\infty, 0]$ that converge exponentially to S as $t \to -\infty$, and are determined uniquely by their trace at time $t = 0$.

By a delicate analysis of the dynamics of an ancient flow using an ODE lemma of Merle–Zaag [14], we were able to obtain a strong characterization for ancient mean curvature flows:

Theorem 3 (Strong uniqueness)**.** *Let S be a closed, smoothly embedded minimal submanifold of a Riemannian manifold $(M, \overline{g})$. There exists an $\varepsilon > 0$ such that if $(\Sigma_t)_{t\leq 0}$ is an ancient mean curvature flow which stays uniformly ε-close to S in the sense of measures, and*

$$\int_{-\infty}^{0} \mathrm{dist}_{\overline{g}}(\Sigma_t, S)\, dt < \infty,$$

then there exists $\tau \geq 0$ so that $(\Sigma_{t-\tau})_{t\leq 0}$ is one of the flows in the I-parameter canonical family.

We briefly remark that this canonical family existence theorem and strong uniqueness theorem do readily give the aforementioned classification results for area pinched ancient mean curvature flows in round n-spheres. Indeed, the compactness theorem of weak mean curvature flows of K. Brakke [4], a modification of the Łojasiewicz–Simon inequality (pioneered by L. Simon in [15]), and the “integrability” of equators and Clifford tori (a concept pioneered by Allard–Almgren [1]), put together guarantee that the decay rate assumption of the strong uniqueness theorem is satisfied, thus giving the necessary canonical classification.

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Classification of 3d κ-solutions

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(joint work with Sigurd Angenent, Simon Brendle, Panagiota Daskalopoulos)

Consider an ancient compact 3-dimensional solution to the Ricci flow

$$\frac{\partial}{\partial} g_{ij} = -2R_{ij} \tag{1}$$

existing for $t \in (-\infty, 0)$ so that it shrinks to a round point at T. The goal is to provide the classification of such solutions under natural geometric assumptions.

In [3], G. Perelman established the existence of a rotationally symmetric ancient κ-noncollapsed solution on S^3 which is not a soliton. This is a type II ancient solution backward in time, namely its scalar curvature satisfies $\sup_{M\times(-\infty,0)} |t||R(x,t)| = \infty$ and forms a type I singularity forward in time, since it shrinks to a round point. Perelman's ancient solution has backward in time limits which are the Bryant soliton and the round cylinder $S^2 \times \mathbb{R}$, depending on how the sequence of points and times about which one rescales are chosen. These are the only backward in time limits of the Perelman ancient solution. Let us remark that the Perelman ancient solution is *noncollapsed*.

The well known Hamilton-Ivey pinching estimate tells us that any two or three dimensional Ricci flow ancient solution, with bounded curvature at each time slice, has nonnegative sectional curvature. Since our solution $(S^3, g(t))$ is closed, the strong maximum principle implies that the sectional curvatures, and hence the entire curvature operator, are strictly positive. It follows by Hamilton's Harnack estimate that $R_t \geq 0$, yielding the existence of a uniform constant $C > 0$ so that $\|\mathrm{Rm}\|_{g(t)} \leq C$, for all $t \in (-\infty, t_0)$. The above discussion yields that any closed 3-dimensional κ-noncollapsed ancient solution is actually a *κ-solution*, in the sense that was defined by Perelman in [3].

In a joint work with Brendle and Daskalopoulos we show the following result that was conjectured by Perelman in [3].

Theorem 1 (Brendle, Daskalopoulos, Sesum). *Let $(S^3, g(t))$ be a compact, ancient κ-noncollapsed solution to the Ricci flow* (1) *on S^3. Then $g(t)$ is either a family of contracting spheres or Perelman's solution.*