These comparisons are in general more strong then Alexandrov comparison. It turns out that some of these comparisons are closely related to certain properties of Riemannian manifolds, arising in optimal transport theory. These are Ma-Trudinger-Wang (MTW) condition and convex tangent injectivity domain property (CTIL). These two properties are in some sense almost equivalent to the transport continuity property (TCP). We show that a Riemannian manifold satisfies $(4,1)$-bipolar comparison if and only if it is (CTIL) and ($\mathrm{MTW}^{\not\angle}$), where ($\mathrm{MTW}^{\not\angle}$) is some stronger version of (MTW). We conjecture that $(4,1)$-bipolar comparison implies (TCP) or may be even equivalent to (TCP).

The spaces satisfying bipolar comparisons for all $k,l$ include all subspaces of quotients of Hilbert spaces by groups of isometries. The class of such quotients includes all double quotients of compact Lie groups with bi-invariant metrics (by Terng–Thorbergsson-95).

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## Minimal surfaces and the Allen–Cahn equation on 3-manifolds


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(joint work with Otis Chodosh)


Fix $(M^3, g)$ to be a closed Riemannian 3-manifold. The Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1}$$

is a semilinear PDE which is deeply linked to the theory of minimal hypersurfaces. For instance, it is known that the Allen–Cahn functional

$$E_\varepsilon[u] := \int_M \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) d\mu_g,$$

whose critical points satisfy (1), Γ-converges as $\varepsilon \to 0$ to the perimeter functional [12, 14] and the level sets of $E_\varepsilon$-minimizing solutions to (1) converge as $\varepsilon \to 0$ to area-minimizing boundaries. When $u$ is not $E_\varepsilon$-minimizing, the limit may occur with high multiplicity. Together with Otis Chodosh we studied solutions to (1) on 3-manifolds with uniform $E_\varepsilon$-bounds and uniform Morse index bounds and showed that multiplicity does *not* occur when the metric $g$ is "bumpy," i.e., when no immersed minimal surface carries nontrivial Jacobi fields; bumpy metrics are generic in the sense of Baire category—see White [16]. This resolves a strong form of the "multiplicity one" conjecture of Marques–Neves [10] for Allen–Cahn. Our main theorem is:

**Theorem 1** ([1]). *Suppose that $u_i$ are critical points of $E_{\varepsilon_i}$ with $\varepsilon_i \to 0$ and*

$$E_{\varepsilon_i}[u_i] \leq E_0, \ \operatorname{ind}(u_i) \leq I_0 \textit{ for all } i = 1, 2, \ldots$$

*Passing to a subsequence, for each $t \in (-1,1)$, $\{u_i = t\}$ converges in the Hausdorff sense and in $C^{2,\alpha}_{\mathrm{loc}}$ away from $\leq I_0$ points to a smooth closed minimal surface $\Sigma$. For any connected component $\Sigma' \subset \Sigma$, either:*

- *$\Sigma'$ is two-sided and occurs as a multiplicity one graphical $C^{2,\alpha}$ limit; or,*
- *$\Sigma'$ is a two-sided stable minimal surface with a positive Jacobi field and which occurs as multiplicity two limit or higher; or*
- *$\Sigma'$ is one-sided and its two-sided double cover is a stable minimal surface with a positive Jacobi field.*

The case $I_0 = 0$ of Theorem 1 is largely a consequence of Theorem 2 below:

**Theorem 2** ([1]). *Let $u$ be a stable critical point of $E_\varepsilon$. As $\varepsilon \to 0$ we have*

$$\exp(-\sqrt{2}\varepsilon^{-1} \operatorname{dist}_g(x, x')) = o(\varepsilon^2 |\log \varepsilon|), \tag{2}$$

*for any $x$, $x'$ that belong to different connected components of $\{u = 0\}$, as well as*

$$\|H\|_{C^0(\{u=t\})} = o(\varepsilon |\log \varepsilon|), \tag{3}$$

$$\|A\|_{C^{0,\alpha}(\{u=t\})} = O(1) \tag{4}$$

*for the mean curvature $H$ and the second fundamental form $A$ of $\{u = t\}$, $|t| \leq 1 - \beta$. Here $\alpha$, $\beta \in (0,1)$, and the constants depend on $\alpha$, $\beta$, $E_\varepsilon[u]$, $M$, $g$.*

This theorem is based on sharpening some recent novel work of Wang–Wei [15] that resolved the "finite index implies finitely many ends" Allen–Cahn conjecture without energy bounds in $\mathbf{R}^2$. Their remarkable insight was the reduction of the question of regularity to a question about the distance among the sheets comprising $\{u = 0\}$. We remark that (2)-(3) are stronger than the bounds obtained in [15] (on the other hand, we have dependence on $E_\varepsilon[u]$). Though bounds on the order of those in [15] suffice for obtaining $C^{2,\alpha}$ estimates for *some* $\alpha \in (0,1)$, the stronger bounds stated in (2)-(3) ensure that the level sets are mean curvature dominated and play a crucial role for our geometric applications.

Our other result, which is proved for all ambient dimensions, is a resolution of the "index lower bound" conjecture of Marques–Neves [10] (for Allen–Cahn):

**Theorem 3** ([1]). *Suppose that $u_i$ are critical points of $E_{\varepsilon_i}$ with $\varepsilon_i \to 0$ and $\{u_i = t\}$ converging to a smooth two-sided minimal hypersurface $\Sigma$ with multiplicity one. Then, after possibly passing to a subsequence,*

$$\operatorname{ind}(\Sigma) \leq \lim_i \operatorname{ind}(u_i) \tag{5}$$

$$\leq \lim_i [\operatorname{ind}(u_i) + \operatorname{nul}(u_i)] \leq \operatorname{ind}(\Sigma) + \operatorname{nul}(\Sigma). \tag{6}$$

Only the last inequality, (6), is new to [1]; (5) was previously shown to be true by Gaspar [4] without two-sidedness or multiplicity one assumptions. (See also the works of Hiesmayr [7], Le [13].) To prove (6), one needs to obtain a very precise understanding of how $\{u_i = t\}$ tend to $\Sigma$. To do so, we borrow ideas from [2] where the authors studied the index and nullity for solutions of (1) that they constructed; an added difficulty is that in our case the $u_i$ are essentially arbitrary.

Finally we point out an important consequence of this work to the study of minimal surfaces in closed Riemannian 3-manifolds $(M^3, g)$. Recall the following minmax construction of Gaspar–Guaraco (which works for ambient dimensions $3 \leq n \leq 7$):

**Theorem 4** (Gaspar–Guaraco [5]). *Let $p \in \{1, 2, 3, \ldots\}$. For all sufficiently small $\varepsilon > 0$, there exists a critical point $u_{\varepsilon,p}$ of $E_\varepsilon$ with*

$$E_\varepsilon[u_{\varepsilon,p}] \sim p^{1/3}, \text{ and} \tag{7}$$

$$\operatorname{ind}(u_{\varepsilon,p}) \leq p \leq \operatorname{ind}(u_{\varepsilon,p}) + \operatorname{nul}(u_{\varepsilon,p}). \tag{8}$$

Let us now assume that the metric $g$ on $M$ is bumpy, i.e., that there are no closed immersed minimal surfaces that carry nontrivial Jacobi fields. Invoking Theorems 1, 3, 4, together with the bumpiness condition and Ejiri–Micallef [3], we obtain a closed embedded minimal hypersurface $\Sigma_p$ with

$$|\Sigma_p| \sim p^{1/3}, \ \operatorname{ind}(\Sigma) = p, \ \operatorname{genus}(\Sigma_p) \geq \tfrac{1}{6}p - O(p^{1/3}). \tag{9}$$

In particular, we resolve a conjecture due to Yau [17] in the case of bumpy metrics:

**Corollary 1.** *Any closed $(M^3, g)$ with a bumpy metric $g$ contains infinitely many closed embedded minimal surfaces. They satisfy* (9)*.*

Irie–Marques–Neves [8] previously resolved Yau's conjecture in a Baire-generic sense using the Liokumovich–Marques–Neves Weyl law for the Almgren–Pitts width spectrum [9]. See also the more recent work of Gaspar–Guaraco [6] who proved the Weyl law in the Allen–Cahn setting and obtained similar conclusions as [8]. Our corollary also carries through when $\operatorname{Ric}_g > 0$; see also the previous work of Marques–Neves [11] using Almgren–Pitts in this same setting.

**Remark**. Theorems 2-3 are stated for closed manifolds, i.e., those without boundary, but there are analogs in case $\partial M \neq \emptyset$. See [1].

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# Ricci Flow on Cohomogeneity one manifolds

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We study the Ricci flow in the setting of cohomogeneity one manifolds, i.e. a Riemannian manifold $(M, g)$ with a group $G$ acting isometrically such that the orbit space $M/G$ is one-dimensional. Since isometries are preserved under the flow, the evolving metrics are also invariant. Heuristically this means the Ricci flow reduces to a system of coupled PDEs in 2 variables, one for time and one for space. This makes its analysis much more tractable. In several works including [1], [2], [4] and [5], this structure has been utilized to obtain new information about the Ricci flow. In joint work with R. G. Bettiol [3] we used this framework to exhibit the first examples of compact 4-manifolds with metrics of nonnegative sectional