

YI WANG - MATH 258
HARMONIC ANALYSIS AND ISOPERIMETRIC INEQUALITIES

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1. MAXIMAL FUNCTIONS

We will be working on $\mathbb{R}^n$ and we will denote Lebesgue measure by m .

Definition 1.1. Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The maximal function is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dy.$$

Example 1.2. If f is continuous with compact support on $\mathbb{R}^n$, then $Mf(x) \sim C|x|^{-n}$ for $|x| \gg 1$.

Definition 1.3. Suppose μ is a measure (ordinary, signed, or complex). Its maximal function is defined as

$$M\mu(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |d\mu|(y).$$

Example 1.4. If δ_0 is the Dirac measure at $0 \in \mathbb{R}^n$, then $M\delta_0(x) = C_n|x|^{-n}$ for all $x \in \mathbb{R}^n$.

Maximal functions come up in the study of convolutions, as the following estimate suggests.

Proposition 1.5. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\phi \geq 0$ smooth, radially symmetric and decreasing. Then

$$|(f * \phi)(x)| \leq C_n A Mf(x),$$

where

$$A = \int_{\mathbb{R}^n} \phi(x) dx.$$

Proof. It suffices to show this for $x = 0$. Using n -dimensional spherical coordinates,

$$\begin{aligned} (f * \phi)(0) &= \int_{\mathbb{R}^n} f(y) \phi(0 - y) dy = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r\theta) \phi(r) r^{n-1} d\theta dr \\ &= \int_0^\infty \Lambda'(r) \phi(r) dr = - \int_0^\infty \Lambda(r) \phi'(r) dr \end{aligned}$$

where

$$\Lambda(R) \triangleq \int_0^R \int_{\mathbb{S}^{n-1}} f(r\theta) r^{n-1} d\theta dr = \int_{|x| \leq R} f(x) dx.$$

By the definition of the maximal function, $|\Lambda(R)| \leq C_n R^n Mf(0)$. Then

$$|(f * \phi)(0)| \leq \int_0^\infty C(n) r^n Mf(0) \phi'(r) dr = C_n Mf(0) \int_0^\infty r^n \phi'(r) dr = C'_n A Mf(0),$$

having integrated by parts in the last step. $\square$

Proposition 1.6. *The following are true:*

- (1) *If $f \in L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$, then $Mf(x)$ is finite for a.e. x .*
- (2) *If $f \in L^1(\mathbb{R}^n)$ then $Mf \in L^1_w(\mathbb{R}^n)$, and*

$$m\{Mf > \alpha\} \leq C_n \alpha^{-1} \|f\|_{L^1}.$$

- (3) *If $f \in L^p(\mathbb{R}^n)$ for $p \in (1, \infty]$ then $Mf \in L^p$, and*

$$\|Mf\|_{L^p} \leq C_{n,p} \|f\|_{L^p}.$$

Remark 1.7. Clearly (1) follows from (2), (3), so we can just prove those. Also, note that we cannot improve (2) to show $Mf \in L^1$. If f is continuous and compactly supported, then of course it is L^1 but $Mf(x) \sim C_n |x|^{-n}$ is not L^1 .

Proof of (2). Let $E_\alpha = \{Mf > \alpha\}$. Then for all $x \in E_\alpha$ there exists $r_x > 0$ such that

$$\int_{B(x, r_x)} |f(y)| dy > \alpha m(B(x, r_x)).$$

The balls $B(x, r_x)$ cover E_α . Note that the radii r_x are uniformly bounded:

$$r_x \leq \left(\frac{1}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy \right)^{1/n}.$$

By the 5-covering lemma there exists a sequence of disjoint balls $B_1, B_2, \dots$ among the $B(x, r_x)$ such that $\cup_i 5B_i$ covers E_α . Then

$$5^n \sum_i m(B_i) \geq m(E_\alpha).$$

At the same time, by our construction of r_x we have

$$5^n \sum_i m(B_i) \leq \frac{5^n}{\alpha} \sum_i \int_{B(x_i, r_{x_i})} |f(y)| dy \leq \frac{5^n}{\alpha} \|f\|_{L^1},$$

and the result follows. □

Proof of (3). We use the following trick; define

$$f_1(x) \triangleq \begin{cases} f(x) & \text{if } |f(x)| > \alpha/2 \\ 0 & \text{else} \end{cases}$$

Then $|f(x)| \leq |f_1(x)| + \alpha/2$, so $Mf(x) \leq Mf_1(x) + \alpha/2$, which in turn gives that

$$E_\alpha = \{Mf > \alpha\} \subseteq \{Mf_1 > \alpha/2\}.$$

By (2) on f_1 ,

$$m(E_\alpha) \leq \frac{2 \cdot 5^n}{\alpha} \|f_1\|_{L^1} = \frac{2 \cdot 5^n}{\alpha} \int_{|f| > \alpha/2} |f(x)| dx.$$

On the other hand we also have

$$\int_{\mathbb{R}^n} |Mf(x)|^p dx = \int_0^\infty p \alpha^{p-1} m(E_\alpha) d\alpha \leq \int_0^\infty p \alpha^{p-1} \frac{2 \cdot 5^n}{\alpha} \int_{|f| > \alpha/2} |f(x)| dx d\alpha$$

$$= C_{n,p} \int_{\mathbb{R}^n} \left(\int_0^{2|f(x)|} \alpha^{p-2} d\alpha \right) |f(x)| dx \leq C'_{n,p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

which implies the required estimate. $\square$

Corollary 1.8 (Differentiation theorem). *For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r \downarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy = f(x).$$

Proof. Let

$$\Omega f(x) \triangleq \limsup_{r \downarrow 0} f_r(x) - \liminf_{r \downarrow 0} f_r(x)$$

where $f_r(x)$ is the mean value of f on $B(x,r)$. If $g \in \mathcal{C}_c^0$ then it is easy to see that $\Omega g \equiv 0$.

If $f \in L^1_{\text{loc}}$ and $\varepsilon > 0$ then it is not hard to see that inside a ball $B \subset \mathbb{R}^n$ we can decompose $f = g + h_\varepsilon$ where $g \in \mathcal{C}_c^0(B)$ and $\|h_\varepsilon\|_{L^1(B)} \leq \varepsilon$. Since $\Omega g \equiv 0$, we just need to understand Ωh_ε . But

$$\Omega h_\varepsilon(x) \leq 2 M h_\varepsilon(x)$$

so

$$m\{\Omega h_\varepsilon > \alpha\} \leq m\{M h_\varepsilon > \alpha/2\} \leq \frac{2 \cdot 5^n}{\alpha} \|h_\varepsilon\| \rightarrow 0$$

as $\varepsilon \downarrow 0$. Since $\varepsilon > 0$ were arbitrary, $\Omega f(x) = 0$ a.e. x . The fact that the limit is in fact what we expect it to be follows from the continuity of $h \mapsto \int f(\cdot + h)$. $\square$

2. CALDERON-ZYGMUND DECOMPOSITION

The Calderon-Zygmund decomposition, as we will see below, is simultaneously a decomposition of functions as well as a decomposition of sets.

Theorem 2.1. *Let $f \in L^1(\mathbb{R}^n)$, $f \geq 0$, and $\alpha > 0$. Then we can decompose $\mathbb{R}^n$ so that*

- (1) $\mathbb{R}^n = F \cup \Omega$, for disjoint sets F closed, Ω open,
- (2) $f \leq \alpha$ a.e. on F ,
- (3) $\Omega = \cup_k Q_k$ for closed cubes Q_k whose interiors are disjoint, and

$$a < \frac{1}{m(Q_k)} \int_{Q_k} f \leq 2^n \alpha$$

Corollary 2.2. *We can decompose $f \geq 0$ above into a "good part" (g) and a "bad part" (b), i.e. $f = g + b$ where*

$$g(x) = \begin{cases} f(x) & \text{for } x \in F \\ \frac{1}{m(Q_k)} \int_{Q_k} f & \text{for } x \in Q_k \end{cases}$$

and

$$b(x) = f(x) - g(x).$$

The "good part", g , is bounded:

$$0 \leq g(x) \leq 2^n \alpha.$$

The "bad part" has mean value zero on each Q_k :

$$\int_{Q_k} b = 0.$$

Proof of theorem. Since $f \in L^1$, there exists a large enough diameter such that

$$\frac{1}{m(Q')} \int_{Q'} f \leq \alpha$$

for all cubes Q' of said diameter. Decompose $\mathbb{R}^n$ into a mesh of such cubes. Fixing such a cube Q' , inductively divide it into 2^n equal cubes Q'' . Certainly for each of them either

$$\frac{1}{m(Q'')} \int_{Q''} f \leq \alpha \text{ or } \frac{1}{m(Q'')} \int_{Q''} f > \alpha$$

is true. If the latter is true, stop subdividing the cube. At that stage it's true that

$$\frac{1}{m(Q'')} \int_{Q''} f \leq \frac{1}{m(Q'')} \int_{Q'} f \leq \frac{2^n}{m(Q')} \int_{Q'} f \leq 2^n \alpha.$$

Have Q'' be one of the Q_k of Ω . Notice that only countably many such Q_k get constructed. Then set $F = \mathbb{R}^n \setminus \Omega$. By the differentiation theorem, $f \leq \alpha$ on F . $\square$

Corollary 2.3. *There exist A, B depending on n such that, for $f, \alpha, F, \Omega, Q_k$ as above,*

- (1) $m(\Omega) \leq \frac{A}{\alpha} \|f\|_{L^1}$, and
- (2) $\frac{1}{m(Q_k)} \int_{Q_k} f \leq B \alpha$.

Proof. This is a trivial consequence of the theorem with $A = 1, B = 2^n$. $\square$

There is, however, an alternative proof of the corollary that provides better understanding for the sets F, Ω . It depends on the following lemma whose proof we skip:

Lemma 2.4. *Suppose F is closed and nonempty. Then*

- (1) *we can decompose its complement $\Omega = \mathbb{R}^n \setminus F$ as $\Omega = \cup_k Q_k$, for closed cubes Q_k whose interiors are disjoint, and*
- (2) *$c \operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, F) \leq c \operatorname{diam}(Q_k)$ for a fixed constant $c = c_n$.*

Alternative proof of corollary. The correct way to think of F, Ω is as

$$F \triangleq \{Mf \leq \alpha\} \text{ and } \Omega \triangleq \{Mf > \alpha\}.$$

These sets are closed and open, respectively, because Mf is lower semicontinuous.

Part (1) follows from the main theorem we proved on maximal functions, according to which $m(\Omega) \leq \frac{5^n}{\alpha} \|f\|_{L^1}$.

Part (2) follows from the lemma above. Fix Q_k in the decomposition of Ω , and let $p_k \in F$ be closest to Q_k . Choose B_k to be the smallest ball centered at p_k which covers Q_k . Then

$$\alpha \geq Mf(p_k) \geq \frac{1}{m(B_k)} \int_{B_k} f \geq \frac{1}{m(B_k)} \int_{Q_k} f \geq \frac{c(n)}{m(Q_k)} \int_{Q_k} f$$

which is the required result. $\square$

3. INTERPOLATION OF L^p SPACES

In this section we treat the Marcinkiewicz interpolation theorem. We need to introduce some notation before stating the result.

Definition 3.1. An operator $T : L^p(\mathbb{R}^n) \rightarrow L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be of

- (1) (strong) type (p, q) , for $p, q \in [1, \infty]$, provided $\|Tf\|_{L^q} \leq C \|f\|_{L^p}$ for all $f \in L^p$,
- (2) weak type (p, q) , for $p \in [1, \infty]$, $q \in [1, \infty)$, provided $m\{|Tf| > \alpha\} \leq \left(\frac{A}{\alpha} \|f\|_{L^p}\right)^q$ for all $f \in L^p$,
- (3) weak type (p, ∞) if it is of (strong) type (p, ∞) .

Remark 3.2. The definition of weak (p, ∞) is in line with what you would obtain from weak types (p, q) by setting $q = \infty$. If $f \in L^p$ then for $\alpha \gg 1$ we have $\frac{A}{\alpha} \|f\|_{L^p} < 1$, so $\left(\frac{A}{\alpha} \|f\|_{L^p}\right)^\infty = 0$ and therefore $|Tf| \leq \alpha$ a.e.

Definition 3.3. Let $p_1, p_2 \in [1, \infty]$ be given. We denote by $L^{p_1} + L^{p_2}$ the set of functions f that can be decomposed as $g + h$ for $g \in L^{p_1}$, $h \in L^{p_2}$.

Theorem 3.4 (Marcinkiewicz interpolation). *Suppose $r \in (1, \infty]$, and that $T : L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n) \rightarrow L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies*

- (1) $|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|$,
- (2) T is of weak type $(1, 1)$, and
- (3) of weak type (r, r) .

Then T is of (strong) type (p, p) for all $p \in (1, r)$.

Remark 3.5. It is implicit that we need to check that $L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$. Indeed we can decompose $f \in L^p$ as $f_1 + f_2$ where

$$f_1 = \begin{cases} f & \text{if } |f| > \gamma \\ 0 & \text{else} \end{cases}$$

and

$$f_2 = \begin{cases} 0 & \text{if } |f| > \gamma \\ f & \text{else.} \end{cases}$$

It's not hard to see that $f_1 \in L^1$ and $f_2 \in L^r$.

Proof. We treat the case $r < \infty$; the infinite case only requires minor modifications. Decompose $f = f_1 + f_2$ as above for $\gamma = \alpha$. Since

$$|Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)|$$

we have

$$\{|Tf| > \alpha\} \subset \{|Tf_1| > \alpha/2\} \cup \{|Tf_2| > \alpha/2\}$$

and therefore

$$m\{|Tf| > \alpha\} \leq m\{|Tf_1| > \alpha/2\} + m\{|Tf_2| > \alpha/2\}$$

$$\begin{aligned}
&\leq \frac{A_1}{\alpha/2} \|f_1\|_{L^1} + \left(\frac{A_r}{\alpha/2} \|f_2\|_{L^r} \right)^r \\
&= \frac{2A_1}{\alpha} \int_{|f|>\alpha} |f| + \left(\frac{2A_r}{\alpha} \right)^r \int_{|f|\leq\alpha} |f|^r.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^n} |Tf|^p &= p \int_0^\infty \alpha^{p-1} m\{|Tf| > \alpha\} d\alpha \\
&\leq p \int_0^\infty \alpha^{p-1} \frac{2A_1}{\alpha} \int_{|f|>\alpha} |f(x)| dx d\alpha + p \int_0^\infty \alpha^{p-1} \frac{(2A_r)^r}{\alpha^r} \int_{|f|\leq\alpha} |f(x)|^r dx d\alpha \\
&= C_{n,p} \int_{\mathbb{R}^n} \int_0^{|f(x)|} \alpha^{p-2} d\alpha |f(x)| dx + C_{n,p} \int_{\mathbb{R}^n} \int_{|f(x)|}^\infty \alpha^{p-1-r} d\alpha |f(x)|^r dx \\
&= C_{n,p} \int_{\mathbb{R}^n} |f|^p
\end{aligned}$$

□

Remark 3.6. By carefully keeping track of the constants above we get:

$$\|Tf\|_{L^p} \leq C \|f\|_{L^p} \text{ with } C = \left(p \left(\frac{2A_1}{p-1} + \frac{(2A_r)^r}{r-p} \right) \right)^{1/p}.$$

What we've done here provides an alternative proof of the maximal function theorem using Calderon-Zygmund decompositions.

Corollary 3.7. *We know that the maximal function operator is of weak type $(1, 1)$ and (∞, ∞) . By the Marcinkiewicz theorem, it is also of type (p, p) .*

4. SINGULAR INTEGRALS

Equipped with what we've developed so far, we can start to understand operators defined by singular integrals. As a motivating example, we consider the **Hilbert transform**, defined as follows:

Definition 4.1. Let f be a real-valued function, the Hilbert transform is defined to be

$$Hf \equiv \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy \quad (= \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} dy)$$

whenever the right-hand side exists.

We shall see later that H extends to a bounded linear operator from L^p to itself for $1 < p < \infty$. Before going into that, let's fix some notations and recall the basic facts about convolutions and the Fourier transform on $\mathbb{R}^n$.

Below $C_0(\mathbb{R}^n)$ will denote the space of continuous functions that vanish at infinity. $\mathcal{B}(\mathbb{R}^n)$ will denote the space of finite Radon measures on $\mathbb{R}^n$, which is the dual space of $C_0(\mathbb{R}^n)$. Recall that the norm on $\mathcal{B}(\mathbb{R}^n)$ is given by

$$\|\mu\| = \int_{\mathbb{R}^n} d|\mu|$$

Moreover, by considering the correspondence

$$f \mapsto f d\mu$$

we can identify the space $L^1(\mathbb{R}^n)$ with the subspace of $\mathcal{B}(\mathbb{R}^n)$ consisting of measures absolutely continuous w.r.t the Lebesgue measure. This is a proper subspace since the Dirac δ -measure is not absolutely continuous.

As is the case of L^1 functions, we can convolve measures in $\mathcal{B}$ with one another. Specifically, we make the following definition.

Definition 4.2. Given two measure μ_1 and μ_2 in $\mathcal{B}$, their convolution $\mu = \mu_1 * \mu_2$ is given by

$$\mu(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x+y) d\mu_1(x) d\mu_2(y), \quad \forall f \in C_0(\mathbb{R}^n)$$

Remark 4.3.

- (1) By changing the order of integration and switching x and y , we see that $\mu_1 * \mu_2 = \mu_2 * \mu_1$.
- (2) We have the following inequality which guarantees that $\mu_1 * \mu_2$ is again in $\mathcal{B}$:

$$\|\mu_1 * \mu_2\| \leq \|\mu_1\| \|\mu_2\|$$

Definition 4.4. Given a $f \in L^1(\mathbb{R}^n)$ and $\mu \in \mathcal{B}(\mathbb{R}^n)$, define their Fourier transform to be

$$\begin{aligned} \mathcal{F}(f)(\xi) &= \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx \\ \mathcal{F}(\mu) &= \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} d\mu(x) \end{aligned}$$

We summarize the important properties of $\mathcal{F}$ as the following proposition:

Proposition 4.5.

- (1) Let $f \in L^1(\mathbb{R}^n)$, then $\mathcal{F}(f)$ is in $C_0(\mathbb{R}^n)$. Moreover, given another function $g \in L^1(\mathbb{R}^n)$, we have

$$\widehat{(f * g)} = \hat{f} \hat{g}.$$

In other words, $\mathcal{F}$ takes convolutions into usual products in the phase space.

- (2) (The Plancherel identity) Let $f \in L^1 \cap L^2(\mathbb{R}^n)$, then

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

Thus, by continuity, $\mathcal{F}$ extends to an isometry from $L^2(\mathbb{R}^n)$ to itself.

- (3) (Extension of the convolution property) Let $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Define $h = f * g$. Then

$$\hat{h} = \hat{f} \hat{g}$$

Next we look at convolution operators and state some important facts about them.

Fact 4.6. (Convolution commutes with translation) Fix $y \in \mathbb{R}^n$ and define the operator τ_y by $\tau_y f(x) = f(x + y)$. Moreover, take $\mu \in \mathcal{B}(\mathbb{R}^n)$ and define an operator $T : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ by

$$Tf = f * \mu$$

Then we have

$$\tau_y T = T \tau_y$$

Fact 4.7. (Characterization of bounded linear maps on L^1 that commute with translations) Let T be a bounded linear map from $L^1(\mathbb{R}^n)$ to itself that commutes with any translation. Then there exists a measure $\mu \in \mathcal{B}(\mathbb{R}^n)$ such that

$$Tf = f * \mu, \quad \forall f \in L^1(\mathbb{R}^n).$$

Moreover, the operator norm of T is equal to $\|\mu\|$.

Fact 4.8. (Characterization of bounded linear maps on L^2 that commute with translations) Let T be a bounded linear map from $L^2(\mathbb{R}^n)$ to itself that commutes with any translation. Then there exists a function $m \in L^\infty(\mathbb{R}^n)$ such that

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi), \quad \forall f \in L^2(\mathbb{R}^n).$$

Moreover, $\|T\| = \|m\|_{L^\infty}$.

Remark 4.9. The function m in the last fact is called a **multiplier**.

Now we make our first step towards understanding operators defined by singular integrals.

Theorem 4.10. Let $K \in L^2(\mathbb{R}^n)$ and suppose

- (1) $\|\hat{K}\|_{L^\infty} \leq B$
- (2) $K \in C^1(\mathbb{R}^n - \{0\})$ and $|DK(x)| \leq C|x|^{-n-1}$

For $f \in L^1 \cap L^p(\mathbb{R}^n)$, define $Tf = K * f$. Then for all $1 < p < \infty$, there is a constant $A = A(p, B, n, C)$ such that

$$\|Tf\|_{L^p} \leq A\|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n)$$

Remark 4.11.

- (1) The constant A does not depend on the L^2 -norm of the kernel K .
- (2) The Hilbert transform is not covered by this theorem since the kernel, $1/y$, fails the first condition.

Proof of theorem 4.10. The idea is to apply the Marcinkiewicz interpolation theorem followed by a duality argument. Accordingly, the proof will be carried out in three steps.

Step 1 (T is weak $(2, 2)$)

This step is easy. Since $\|\hat{K}\|_{L^\infty} \leq B$ and $\widehat{Tf} = \hat{K}\hat{f}$, by Plancherel identity we have

$$\|Tf\|_{L^2} = \|\widehat{Tf}\|_{L^2} \leq B\|\hat{f}\|_{L^2} = B\|f\|_{L^2}, \quad \forall f \in L^1 \cap L^2(\mathbb{R}^n)$$

Hence, by continuity, T extends to L^2 and is strong $(2, 2)$. Thus T is weak $(2, 2)$.

Step 2 (T is weak $(1, 1)$)

This step is where most of the work goes. We want to establish an estimate of the form:

$$m(\{|Tf| > \alpha\}) \leq \frac{C}{\alpha} \|f\|_{L^1}, \quad \forall \alpha$$

Note that this would be easy if we know that $\|f\|_\infty \leq \alpha$, for then we deduce that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} |f|^2 dx \leq \alpha \|f\|_{L^1}$$

Using the result of step 1, we can proceed as follows to get the desired estimate.

$$m(\{|Tf| > \alpha\}) \leq \frac{1}{\alpha^2} \|Tf\|_{L^2}^2 \leq \frac{B^2}{\alpha^2} \|f\|_{L^2}^2 \leq \frac{B^2}{\alpha} \|f\|_{L^1}$$

This suggests applying the Calderon-Zygmund decomposition to f at the level α . Then we get:

- (1) $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$
- (2) $|f(x)| \leq \alpha$ on F , and

$$\alpha < \frac{1}{m(Q_j)} \int_{Q_j} |f| dx \leq 2^n \alpha, \quad (\Omega = \cup_j Q_j^{\text{int}})$$

- (3) There is a constant C such that

$$C^{-1} \text{dist}(Q_j, F) \leq \text{diam}(Q_j) \leq C \text{dist}(Q_j, F) \text{ for each } Q_j.$$

Furthermore, we also get a decomposition $f(x) = g(x) + b(x)$, where

$$(1) \quad g(x) = \begin{cases} f(x) & \text{on } F \\ \frac{1}{m(Q_j)} \int_{Q_j} f dx & \text{on } Q_j \end{cases}, \quad b(x) = \begin{cases} 0 & \text{on } F \\ f(x) - \frac{1}{m(Q_j)} \int_{Q_j} f dx & \text{on } Q_j \end{cases}$$

- (2) $\|g\|_{L^\infty} \leq 2^n \alpha$, $\int_{Q_j} b dx = 0$ on each Q_j .

As in the proof of Theorem 3.4, we have

$$m(\{|Tf| > \alpha\}) \leq m(\{|Tg| > \alpha/2\}) + m(\{|Tb| > \alpha/2\})$$

To estimate the first term, we note that

$$\|g\|_{L^2}^2 \leq \|g\|_{L^\infty} \|g\|_{L^1} \leq 2^n \alpha \|g\|_{L^1}$$

Thus, using step 1,

$$m(\{|Tg| > \alpha/2\}) \leq \frac{B^2}{(\alpha/2)^2} \|g\|_{L^2}^2 \leq \frac{2^{n+2} B^2}{\alpha} \|g\|_{L^1} \leq \frac{2^{n+2} B^2}{\alpha} \|f\|_{L^1}.$$

As for the second term, since

$$m(\Omega) = \sum_j m(Q_j) \leq \frac{1}{\alpha} \|f\|_{L^1},$$

it suffices to estimate $m(\{|Tb| > \alpha/2\} \cap F)$.

Fix a j , let $b_j = b|_{Q_j}$ and suppose that Q_j is centered at y_j . Then because b integrates to zero on Q_j , we have

$$Tb_j(x) = \int_{Q_j} K(x-y)b(y)dy = \int_{Q_j} [K(x-y) - K(x-y_j)]b(y)dy \leq \int_{Q_j} \frac{C \operatorname{diam}(Q_j)}{|x-y_j^*|^{n+1}} |b(y)|dy,$$

where we've used the mean value theorem in the last inequality.

Next recall that the cubes Q_j satisfies

$$C^{-1} \operatorname{dist}(Q_j, F) \leq \operatorname{diam}(Q_j) \leq C \operatorname{dist}(Q_j, F) \text{ for each } Q_j.$$

$$\Rightarrow |x - y_j| \approx |x - y_j^*| \approx |x - y| \quad \forall y \in Q_j$$

Therefore

$$\begin{aligned} \int_{Q_j} \frac{C \operatorname{diam}(Q_j)}{|x-y_j^*|^{n+1}} |b(y)|dy &\leq C \frac{\operatorname{diam}(Q_j)}{|x-y_j|^{n+1}} \int_{Q_j} |b_j(y)|dy \\ &\leq C \frac{\operatorname{diam}(Q_j)}{|x-y_j|^{n+1}} m(Q_j) C\alpha \\ &\leq C\alpha \int_{Q_j} \frac{\operatorname{dist}(y, F)}{|x-y|^{n+1}} dy \end{aligned}$$

and hence

$$|Tb(x)| = \left| \sum_j Tb_j(x) \right| \leq C\alpha \int_{\Omega} \frac{\operatorname{dist}(y, F)}{|x-y|^{n+1}} dy$$

Integrating over F , we obtain

$$\begin{aligned} \int_F |Tb(x)|dx &\leq C\alpha \int_F \int_{\Omega} \frac{\operatorname{dist}(y, F)}{|x-y|^{n+1}} dydx \\ &\leq C\alpha \int_{\Omega} \operatorname{dist}(y, F) \int_F |x-y|^{-n-1} dx dy \\ &\leq C\alpha \int_{\Omega} \operatorname{dist}(y, F) \int_{r \geq \operatorname{dist}(y, F)} \omega_n r^{-n-1} r^{n-1} dr dy \\ &\leq C\alpha \int_{\Omega} \operatorname{dist}(y, F) (\operatorname{dist}(y, F))^{-1} dy \\ &\leq C\alpha m(\Omega) \leq C\|f\|_{L^1} \end{aligned}$$

Therefore,

$$\begin{aligned} m(\{|Tb| > \alpha/2\}) &\leq m(\Omega) + m(\{|Tb| > \alpha/2\} \cap F) \\ &\leq m(\Omega) + \frac{C}{\alpha} \|f\|_{L^1} \leq \frac{C}{\alpha} \|f\|_{L^1} \end{aligned}$$

and we've shown that T is weak $(1, 1)$.

Step 3 (Interpolation and duality argument)

Having verified its assumptions, we can apply the Marcinkiewicz interpolation theorem to deduce that

$$\|Tf\|_{L^p} \leq A_p \|f\|_{L^p}, \quad (1 < p < 2).$$

For $2 < p < \infty$, let q be the dual exponent and recall that

$$\|Tf\|_{L^p} = \sup_{\varphi \in L^q, \|\varphi\|_{L^q}=1} \int \varphi Tf dx$$

Given a $\varphi \in L^q$ with $\|\varphi\|_{L^q} = 1$, we have

$$\begin{aligned} \left| \int \varphi(x) Tf(x) dx \right| &= \left| \int \left[\int \varphi(x) K(x-y) f(y) dy \right] dx \right| \\ &= \left| \int \left[\int K(x-y) \varphi(x) dx \right] f(y) dy \right| \\ &= \left| \int f(y) \tilde{T}\varphi(y) dy \right| \leq \|f\|_{L^p} \|\tilde{T}\varphi\|_{L^q} \end{aligned}$$

where $\tilde{T}$ is the convolution operator with kernel $\tilde{K}(x) \equiv K(-x)$.

It's easy to see that $\tilde{K}$ satisfies all the assumptions of Theorem 4.10 with the same constants B and C . Since $1 < q < 2$, by the above arguments,

$$\|\tilde{T}\varphi\|_{L^q} \leq A_q \|\varphi\|_{L^q} = A_q$$

so we have

$$\left| \int \varphi(x) Tf(x) dx \right| \leq A_q \|f\|_{L^p}, \quad \forall \varphi \in L^q, \|\varphi\|_{L^q} = 1$$

Hence for $2 < p < \infty$,

$$\|Tf\|_{L^p} \leq A_q \|f\|_{L^p}$$

Recalling that we've covered the case $p = 2$ in step 1, the proof of the theorem is complete. $\square$

The condition on the decay of $|DK|$ can actually be weakened, as the following corollary shows.

Corollary 4.12. *The conclusion of Theorem 4.10 remains true if we replace*

$$(2) \quad |DK(x)| \leq C|x|^{-n-1}$$

by the following condition

$$(2') \quad \text{There is a constant } B' \text{ such that}$$

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)| dy \leq B', \quad \forall |y| > 0$$

Remark 4.13. (2) implies (2') by the mean value theorem, so (2') is indeed a weaker condition.

Proof of Corollary 4.12.

It suffices to prove the estimate

$$m(\{|Tb| > \alpha/2\}) \leq \frac{C}{\alpha} \|f\|_{L^1}$$

under the present assumptions. Again we apply the Calderon-Zygmund decomposition to f at the level α to get

- (1) $\mathbb{R}^n = F \cup \Omega$, $\Omega = \cup_j Q_j$, Q_j centered at y_j
- (2) $\alpha \leq \frac{1}{m(Q_j)} \int_{Q_j} f(x) dx \leq 2^n \alpha$
- (3) $f = g + b$

For each cube Q_j , consider the cube Q_j^* centered at y_j with

$$\text{diam}(Q_j^*) = 2n^{1/2} \text{diam}(Q_j)$$

Then it's not hard to see that for all $x \notin Q_j^*$,

$$|x - y_j| \geq 2|y - y_j|, \quad \forall y \in Q_j.$$

Now let $\Omega^* = \cup_j Q_j^*$ and $F^* = \mathbb{R}^n - \Omega^*$, then we have

$$m(\Omega^*) \leq (2n^{1/2})^n m(\Omega) \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

Thus, in order to get the desired estimate, it suffices to bound

$$m(\{|Tb| > \alpha/2\} \cap F^*).$$

Following the proof of Theorem 4.10, we can write

$$\begin{aligned} \int_{F^*} |Tb(x)| dx &= \int_{F^*} \sum_j \int_{Q_j} |K(x - y) - K(x - y_j)| |b(y)| dy dx \\ &= \sum_j \int_{Q_j} |b(y)| \int_{F^*} |K(x - y) - K(x - y_j)| dx dy \\ &\leq \sum_j \int_{Q_j} |b(y)| \int_{|x - y_j| \geq 2|y - y_j|} |K(x - y) - K(x - y_j)| dx dy \\ &= \sum_j \int_{Q_j} |b(y)| \int_{|x| \geq 2|y - y_j|} |K(x + y_j - y) - K(x)| dx dy \end{aligned}$$

Applying condition (2'), we arrive at

$$\begin{aligned} \int_{F^*} |Tb(x)| dx &\leq \sum_j \int_{Q_j} |b(y)| CB' dy \leq C \|f\|_{L^1} \\ &\Rightarrow m(\{|Tb| > \alpha/2\}) \leq \frac{C}{\alpha} \|f\|_{L^1}. \end{aligned}$$

□

Next we prove another result on singular integral operators.

Theorem 4.14. *Suppose the kernel K satisfies:*

- (1) $|K(x)| \leq B|x|^{-n}$
- (2) $\int_{|x| \geq 2|y|} |K(x - y) - K(x)| dx \leq B, \quad \forall y$
- (3) $\int_{R_1 < |x| < R_2} K(x) dx = 0, \quad \forall 0 < R_1 < R_2 < \infty$

Given $\varepsilon > 0$, we define

$$T_\varepsilon f = \int_{|y| \geq \varepsilon} f(x-y)K(y)dy$$

Then for all $1 < p < \infty$, there is a constant A_p , independent of ε , such that

$$\|T_\varepsilon f\|_{L^p} \leq A_p \|f\|_{L^p}, \quad \forall \varepsilon.$$

Moreover, for each $f \in L^p$, the limit

$$Tf = \lim_{\varepsilon \rightarrow 0} T_\varepsilon f$$

exists in the strong L^p sense and satisfies

$$\|Tf\|_{L^p} \leq A_p \|f\|_{L^p}$$

The following lemma will be important for the proof of Theorem 4.14.

Lemma 4.15. *Let K be as in the theorem and define*

$$K_\varepsilon(x) = \begin{cases} K(x) & , |x| \geq \varepsilon \\ 0 & , |x| < \varepsilon \end{cases}$$

Then there exists a constant C , independent of ε , such that

$$\sup_{\mathbb{R}^n} |\hat{K}_\varepsilon| \leq C$$

Proof. We first prove the case $\varepsilon = 1$. It's obvious from the definition that K_1 satisfies conditions (1) and (3) in Theorem 4.14. We'll show that (2) holds for K_1 as well. This comes down to handling the following two integrals:

Case 1

$$\int_{|x| \geq 2|y|, |x| \geq 1, |x-y| \leq 1} |K(x)|dx \leq \int_1^2 Br^{-n}r^{n-1}dr \leq CB$$

Notice that we've used condition (1) and the fact that

$$|x| \geq 2|y|, |x-y| \leq 1 \Rightarrow |x| \leq 2$$

Case 2

$$\int_{|x| \geq 2|y|, |x| \leq 1, |x-y| \geq 1} |K(x-y)|dx \leq \int_1^{3/2} Br^{-n}r^{n-1}dr$$

This time we use

$$|x| \geq 2|y|, |x| \leq 1 \Rightarrow |x-y| \leq 3/2$$

Therefore K_1 satisfies condition (2) as well. Now take an arbitrary $y \in \mathbb{R}^n$, then we have

$$\hat{K}_1(y) = \lim_{R \rightarrow \infty} \int_{|x| \leq R} e^{2\pi i x \cdot y} K_1(x) dx$$

$$\begin{aligned}
&= \left(\int_{1 \leq |x| \leq 1/|y|} + \lim_{R \rightarrow \infty} \int_{1/|y| \leq |x| \leq R} \right) e^{2\pi i x \cdot y} K_1(x) dx \\
&= I_1 + I_2
\end{aligned}$$

Using condition (3) (K_1 integrates to zero on annuli centered at 0), we write

$$\begin{aligned}
|I_1| &\leq \left| \int_{1 \leq |x| \leq 1/|y|} [e^{2\pi i x \cdot y} - 1] K_1(x) dx \right| \\
&\leq \int_{1 \leq |x| \leq 1/|y|} C|y||x|B|x|^{-n} dx \\
&\leq CB
\end{aligned}$$

To estimate I_2 , we first notice that by letting $z = \frac{1}{2} \frac{y}{|y|^2}$, we have $e^{2\pi i z \cdot y} = -1$ and so we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} K_1(x) e^{2\pi i y \cdot x} dx &= \frac{1}{2} \int_{\mathbb{R}^n} K_1(x) e^{2\pi i y \cdot x} - K_1(x) e^{2\pi i y \cdot (z+x)} dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} [K_1(x) - K_1(x-z)] e^{2\pi i x \cdot y} dx
\end{aligned}$$

Since the integral I_2 is not over the whole space, we will pick up extra terms when applying the above trick to it. Specifically, we have

$$\begin{aligned}
I_2 &= \frac{1}{2} \lim_{R \rightarrow \infty} \int_{1/|y| \leq |x| \leq R} [K_1(x) - K_1(x-z)] e^{2\pi i x \cdot y} dx \\
&\quad - \frac{1}{2} \int_{1/|y| \leq |x-z|, |x| \leq 1/|y|} K_1(x) e^{2\pi i x \cdot y} dx + \frac{1}{2} \int_{1/|y| \leq |x|, |x-z| \leq 1/|y|} K_1(x) e^{2\pi i x \cdot y} dx
\end{aligned}$$

Since $|z| = \frac{1}{2|y|}$, we apply condition (2) to get a bound on the first term

$$\left| \int_{1/|y| \leq |x|} [K_1(x) - K_1(x-z)] e^{2\pi i x \cdot y} dx \right| \leq B$$

For the second term, note that

$$|x| \leq \frac{1}{|y|}, |x-z| \geq \frac{1}{|y|} \Rightarrow \frac{1}{2|y|} \leq |x| \leq \frac{1}{|y|}$$

Therefore

$$\left| \int_{1/|y| \leq |x-z|, |x| \leq 1/|y|} K_1(x) e^{2\pi i x \cdot y} dx \right| \leq \int_{1/2|y| \leq |x| \leq 1/|y|} B|x|^{-n} dx \leq CB \log 2$$

Similarly

$$\left| \int_{1/|y| \leq |x|, |x-z| \leq 1/|y|} K_1(x) e^{2\pi i x \cdot y} dx \right| \leq \int_{1/|y| \leq |x| \leq 3/2|y|} B|x|^{-n} dx \leq CB \log(3/2)$$

Adding up, we then get

$$|I_2| \leq CB$$

and hence

$$|\hat{K}_1(y)| \leq |I_1| + |I_2| \leq CB$$

Now we show why we may reduce to the case where $\epsilon = 1$. For general $\epsilon > 0$ take $K'(x) = \epsilon^n K(\epsilon x)$. Notice that this function has the same properties as K . Then we have that $K_\epsilon(x) = \epsilon^{-n} K'_1(\epsilon x)$ and hence that

$$|\widehat{K_\epsilon(x)}|(\xi) = |\epsilon^{-n} \widehat{K'_1(\epsilon^{-1}x)}|(\xi) = |\widehat{K'_1}|(\epsilon\xi).$$

Since, by our work above, the right hand side is bounded, then the left hand side must be bounded. $\square$

We'll now apply Lemma 4.15 to prove Theorem 4.14.

Proof. We know that T_ϵ is weak (1,1) map by the results of Lemma 4.15 and by Corollary 4.12. Moreover, we know that the map is bounded on L^2 since $\widehat{K_\epsilon}$ is bounded and since T_ϵ is simply convolution with K_ϵ . Hence, by interpolation we have that T_ϵ is bounded as a map on L^p for all $1 < p \leq 2$. Moreover, since T_ϵ is self-adjoint, we may use duality as in the proof of Theorem 4.10 to get that T_ϵ is bounded as a map on $2 \leq p < \infty$.

It remains to check that the T_ϵ has a limit in L^p as ϵ tends to zero. We'll show this first for $f \in C_c^1(\infty)$. Then we compute

$$\begin{aligned} T_\epsilon f(x) &= \int_{|y| \geq \epsilon} K(y) f(y-x) dy \\ &= \underbrace{\int_{1 \geq |y| \geq \epsilon} K(y) [f(y-x) - f(x)] dy}_{:=I} + \underbrace{\int_{|y| \geq 1} K(y) f(y-x) dy}_{:=II}. \end{aligned}$$

Notice that we use the property that K has mean zero over annuli. Term II is clearly independent of ϵ so we need not worry about it. Hence let's check that the first term has a limit. Since f is compactly support and is C^1 , then it is Lipschitz. Let C_f be the Lipschitz constant of f and we have

$$\int_{1 \geq |y| \geq \epsilon} K(y) [f(y-x) - f(x)] dy \leq C_f \int_{1 \geq |y| \geq \epsilon} K(y) |y| dy \leq C_f B \int_\epsilon^1 \frac{1}{|y|^{n-1}} dy.$$

This last term is bounded independently of ϵ (which can easily be seen by changing to polar coordinates). It is also easy to alter this argument to show that $T_\epsilon f$ is Cauchy in the L^∞ norm. Since term II in $T_\epsilon f$ has compact support (it inherits this property from f), it is then easy to see that $T_\epsilon f$ is Cauchy in L^p . Hence $T_\epsilon f$ tends to an element that we call Tf as ϵ tends to zero.

So now you might believe that it is obvious that $T_\epsilon f$ is Cauchy in L^p for any f . In which case, skip this paragraph; otherwise we'll show it here to be complete. Fix $\delta > 0$ and assume that $g \in L^p$ and find $f \in C_c^1(\mathbb{R}^n)$ such $\|f - g\|_p \leq \delta$. Then choose ϵ_0 small enough such that $\epsilon_1, \epsilon_2 < \epsilon_0$ then $\|T_{\epsilon_1} g - T_{\epsilon_2} g\|_p \leq \delta$. Then we have

$$\|T_{\epsilon_1} f - T_{\epsilon_2} f\| \leq \|T_{\epsilon_1} f - T_{\epsilon_1} g\| + \|T_{\epsilon_2} f - T_{\epsilon_2} g\| + \|T_{\epsilon_1} g - T_{\epsilon_2} g\| \leq A_p \delta + A_p \delta + \delta.$$

Hence $T_\epsilon f$ is Cauchy. $\square$

A direct application of this is to the Hilbert transform. There our kernel is $K(x) = \frac{1}{\pi x}$, and so Theorem 4.14 tells us that the Hilbert transform is well-defined and bounded on L^p for any $1 < p < \infty$.

Singular Integrals that commute with dilation. Define $\tau_\epsilon f(x) = f(\epsilon x)$. Then we look at operators T such that $\tau_\epsilon T \tau_{\epsilon^{-1}} = T$. In the case that T is given by convolution with a kernel, K , then this is the same as the condition

$$K(\epsilon x) = \frac{1}{\epsilon^n} K(x).$$

In other words, when K is homogeneous of order n , or, K can be given by

$$K(x) = \frac{1}{|x|^n} \Omega\left(\frac{x}{|x|}\right),$$

where $\Omega : S^{n-1} \rightarrow \mathbb{R}$. We will abuse notation and write $\Omega(x)$ for any x by simply scaling x to be a unit vector.

Theorem 4.16. *Define*

$$\omega(\delta) := \sup_{\substack{|x-y| \leq \delta, \\ x, y \in S^{n-1}}} |\Omega(x) - \Omega(y)|.$$

Suppose that

- (1) Ω is bounded on S^{n-1} ,
- (2) $\int_{S^{n-1}} \Omega(x) dx = 0$, and
- (3) (Dini-type condition)

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty.$$

Define

$$T_\epsilon f(x) = \int_{|y| \geq \epsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy,$$

and we have that $\|T_\epsilon f\|_p \leq A_p \|f\|_p$ holds for all f and all ϵ . Moreover, we may define $Tf = \lim_\epsilon T_\epsilon f$, where the limit is in L^p . Moreover, for $f \in L^2$ we may write

$$\widehat{Tf}(\xi) = m(\xi) \hat{f}(\xi),$$

where m is homogeneous of degree zero and is, in fact, given by

$$(4.1) \quad m(\xi) = \int_{S^{n-1}} \left[\frac{i\pi}{2} \operatorname{sign}(x \cdot y) - \log |x \cdot y| \right] \Omega(y) dS(y), \quad \text{for } \xi \in S^{n-1}.$$

Proof. Our condition on the integral of Ω gives us that the kernel has mean zero over any annulus. Moreover, we may bound

$$\begin{aligned} \int_{|x| \geq 2|y|} |K(x-y) - K(x)| dx &= \int_{|x| \geq 2|y|} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dx \\ &\leq \underbrace{\int_{|x| \geq 2|y|} \left| \frac{\Omega(x-y) - \Omega(x)}{|x-y|^n} \right| dx}_{:=T_1} + \underbrace{\int \left| \Omega(x) \left[\frac{1}{|x|^n} - \frac{1}{|x-y|^n} \right] \right| dx}_{:=T_2} \end{aligned}$$

We'll first consider T_2 . Here we may use the mean value theorem to get

$$\begin{aligned} T_2 &\leq \int_{|x| \geq 2|y|} \left| \frac{|x|^n - |x-y|^n}{|x|^n |x-y|^n} \right| dx \\ &\lesssim \int_{|x| \geq 2|y|} \frac{|y|^n}{|x|^{n+1} |y|^{n-1}} dx \\ &\lesssim \int_{2|y|}^{\infty} \frac{|y| r^{n-1}}{r^{n+1}} < \infty. \end{aligned}$$

Here we simply used that all norms on $\mathbb{R}^n$ are equivalent, the triangle inequality, and that $|x-y| \geq |x| - |y| \geq |x| - |x|/2$. Now we will consider T_1 , making use of the Dini-type condition. We estimate

$$\begin{aligned} T_1 &\leq \int_{|x| \geq 2|y|} \frac{\omega\left(\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right|\right)}{|x-y|^n} dx \\ &\lesssim \int_{|x| \geq 2|y|} \frac{\omega\left(4\frac{|y|}{|x|}\right)}{4\frac{|x|^n}{|y|^n} |y|^n} dx \\ &\approx \int_{|z| \geq 8} \frac{\omega(|z|^{-1})}{|z|^n} dz \\ &\approx \int_{r \geq 8} \frac{\omega(1/r)}{r^n} r^{n-1} dr \\ &\approx \int_0^{1/8} \frac{\omega(r)}{r} dr < \infty. \end{aligned}$$

Here $\lesssim$ (resp. $\approx$) means less than or equal to (resp. equal to) up to a constant depending only on the dimension.

Hence, our conditions on Ω and the kernel, give us that the hypotheses of Corollary 4.12 are satisfied. Hence we know that T is well-defined. It remains only to show that the Fourier multiplier m exists and is given by the claimed formula. By Fact 4.7, we know that there exists a Fourier multiplier, $m \in L^\infty$. In order to show that m is homogeneous of degree zero, we will use a trick with the dilation operators to show that m is invariant under dilation.

To this end we look at the Fourier transform of T conjugated by τ_ϵ and compute

$$\begin{aligned}\mathcal{F}(\tau_\epsilon T \tau_{\epsilon^{-1}} f)(\xi) &= \epsilon^{-n} \mathcal{F}(T \tau_{\epsilon^{-1}} f)(\epsilon^{-1} \xi) \\ &= \epsilon^{-n} m(\epsilon^{-1} \xi) \mathcal{F}(\tau_{\epsilon^{-1}} f)(\epsilon^{-1} \xi) \\ &= m(\epsilon^{-1} \xi) \mathcal{F}(f)(\xi).\end{aligned}$$

On the other hand, since T commutes with dilation, we know that

$$\mathcal{F}(\tau_\epsilon T \tau_{\epsilon^{-1}} f)(\xi) = \mathcal{F}(T f)(\xi) = m(\xi) \mathcal{F} f(\xi).$$

Hence it follows that $m(\epsilon^{-1} \xi) = m(\xi)$, so that m is homogeneous.

Now we check that m has the form which we claimed. To this end we look at the operator $T_{\epsilon, \eta}$ defined as

$$T_{\epsilon, \eta} f(x) = \int_{\epsilon \leq |y| \leq \eta} \frac{\Omega(y)}{|y|^n} f(x - y) dy.$$

Call $K_{\epsilon, \eta}$ the kernel of this operator. We will show that $\hat{K}_{\epsilon, \eta}$ is bounded independent of ϵ and η and that its limit is given by the formula in (4.1). We will show this limit point wise, but one can check that the limit converges uniformly when $x' \cdot y'$ is bounded away from zero. Hence it follows that $\hat{K}_{\epsilon, \eta}$ converges to m . To that end we rewrite the kernel as

$$\begin{aligned}(4.2) \quad \hat{K}_{\epsilon, \eta}(x) &= \int_{S^{n-1}} \int_{\epsilon}^{\eta} \exp\{2\pi i R r x' \cdot y'\} \Omega(y') \frac{dr}{r} dS(y') \\ &= \int_{S^{n-1}} \int_{\epsilon}^{\eta} [\exp\{2\pi i R r x' \cdot y'\} - \cos(2\pi R r)] \Omega(y') \frac{dr}{r} dS(y') \\ &= \int_{S^{n-1}} I_{\epsilon, \eta}(x', y') \Omega(y') dS(y').\end{aligned}$$

Here we have defined

$$I_{\epsilon, \eta}(x', y') = \int_{\epsilon}^{\eta} [\exp\{2\pi i R r (x' \cdot y')\} - \cos(2\pi R r)] \frac{dr}{r},$$

where we write $x = R x'$ and $y = r y'$ with $|x| = R$ and $|y| = r$. We compute the limit of the imaginary part of this first

$$\text{im}(I_{\epsilon, \eta}) = \int_{\epsilon}^{\eta} \sin(2\pi R r (x' \cdot y')) \frac{dr}{r} \rightarrow \left(\int_0^{\infty} \frac{\sin(t)}{t} dt \right) \text{sign}(x' \cdot y').$$

It is not difficult to show that the integral from ϵ to η is bounded independent of ϵ and η . Using some complex analysis we may evaluate the integral here to obtain

$$\text{im}(I_{\epsilon, \eta}) \rightarrow \frac{\pi}{2} \text{sign}(x' \cdot y').$$

On the other hand, the real part of this is given by

$$\text{Re}(I_{\epsilon, \eta}) = \int_{\epsilon}^{\eta} [\cos(2\pi R r (x' \cdot y')) - \cos(2\pi R r)] \frac{dr}{r} = \int_{2\pi R \epsilon}^{2\pi R \eta} [\cos(tx' \cdot y') - \cos(t)] \frac{dt}{t}.$$

As with above, it is not difficult to show that this integral is bounded independent of ϵ and η . Then we apply Lemma 4.17 to get that

$$\operatorname{Re}(I_{\epsilon,\eta}) \rightarrow -\log |x' \cdot y'|.$$

Putting this together with (4.2), we obtain

$$\hat{K}_{\epsilon,\eta} \rightarrow \int_{S^{n-1}} \left[\frac{i\pi}{2} \operatorname{sign}(x' \cdot y') + \log \frac{1}{|x' \cdot y'|} \right] \Omega(y') dS(y').$$

Since $\hat{K}_{\epsilon,\eta}$ converges as we take ϵ to zero and η to infinite, then $\hat{K}_{\epsilon,\eta}$ converges to m , which implies that m is given by the formula in (4.1). $\square$

Below, for completeness, we include a lemma which Yi alluded to but did not include in the lecture.

Lemma 4.17. *Suppose that h is C^1 , even, mean zero, and periodic with period T . Suppose that $0 < |\mu| \leq 1$. Then*

$$\lim_{\substack{\epsilon \rightarrow 0, \\ \eta \rightarrow \infty}} \int_{\epsilon}^{\eta} [h(\mu r) - h(r)] \frac{dr}{r} = -h(0) \log |\mu|.$$

Proof. We'll assume that $\mu > 0$ without loss of generality. First we compute that

$$\begin{aligned} \int_{\epsilon}^{\eta} [h(\mu r) - h(r)] \frac{dr}{r} &= \int_{\mu\epsilon}^{\mu\eta} \frac{h(r)}{r} dr - \int_{\epsilon}^{\eta} \frac{h(r)}{r} dr \\ &= \int_{\mu\epsilon}^{\epsilon} \frac{h(r)}{r} dr - \int_{\mu\eta}^{\eta} \frac{h(r)}{r} dr \\ &= \int_{\mu\epsilon}^{\epsilon} \frac{h(r) - h(0)}{r} dr - h(0) \log(\mu) - \int_{\mu}^1 \frac{h(r/\eta)}{r} dr \\ &\rightarrow 0 - h(0) \log(\mu) - 0. \end{aligned}$$

The first term tends to zero since h is Lipschitz and the last term tends to zero because h is mean zero. $\square$

5. VECTOR VALUED ANALOGUE

Take $\mathcal{H}$ a separable Hilbert space, and a map $f : \mathbb{R}^n \rightarrow \mathcal{H}$. Define $|f(x)|$ and $\|f\|_{L^p(\mathbb{R}^n)}$ as usual using the Hilbert space norm. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear maps $\mathcal{H}_1 \rightarrow \mathcal{H}_2$.

If $K \in L^q(\mathbb{R}^n, \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$ and $f \in L^p(\mathbb{R}^n, \mathcal{H}_1)$ for p, q conjugate, then

$$g(x) = \int K(x - y) f(y) dy$$

converges in $\mathcal{H}_2$ a.e. x .

Theorem 5.1. *All previous results concerning singular integrals are still valid in the general setting $f : \mathbb{R}^n \rightarrow \mathcal{H}_1$, $K : \mathbb{R}^n \rightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Here $T_{\epsilon}f$ and Tf take values in $\mathcal{H}_2$.*

Recall the Hilbert transform: given $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$Hf(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy$$

having multiplier function

$$m(\zeta) = i \operatorname{sign}(\zeta) = i \frac{\zeta}{|\zeta|}$$

The form of m immediately implies $H^2 = -1$.

Proposition 5.2. *Suppose T is a linear transformation, bounded in $L^2(\mathbb{R})$, and satisfying:*

- (a) T commutes with translations,
- (b) T commutes with positive dilations,
- (c) T anti-commutes with reflections.

Then T is a constant multiple of the Hilbert transform.

Proof. Since T is bounded, linear, and satisfies (a), we have by fact 4.8

$$\mathcal{F}T = m\mathcal{F}$$

for some multiplier function m .

Properties (b) and (c) show that $m(\delta x) = \operatorname{sign}(\delta)m(x)$, as follows:

$$\begin{aligned} \tau_\delta m &= \tau_\delta \mathcal{F}T\mathcal{F}^{-1} \\ &= |\delta|^{-1} \mathcal{F}\tau_{\delta^{-1}}T\mathcal{F}^{-1} \\ &= |\delta|^{-1} \operatorname{sign}(\delta) \mathcal{F}T\tau_{\delta^{-1}}\mathcal{F}^{-1} \\ &= \operatorname{sign}(\delta) \mathcal{F}T\mathcal{F}^{-1} \end{aligned}$$

And hence $m = C \operatorname{sign}$. □

Definition 5.3. Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the Riesz transform Rf is defined by

$$Rf(x) = \frac{1}{\pi |B_1^{n-1}|} \lim_{\epsilon \rightarrow 0} \int_{|y| \geq \epsilon} \frac{y}{|y|^{n+1}} f(x-y) dy$$

Lemma 5.4. *Let $m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be homogenous of degree 0, and suppose m commutes with rotations. Then*

$$m(x) = C \frac{x}{|x|}$$

for some constant C .

Proof. For any $x \neq 0$, we have for an appropriate rotation matrix ρ

$$m(x) = \frac{1}{|x|} m(\rho e_1) = \frac{1}{|x|} \rho m(e_1),$$

where e_1 is the first unit coordinate vector. It suffices to show $m(e_1) = C e_1$. But follows because every rotation ρ fixing e_1 necessarily fixes $m(e_1)$, and (up to scaling) e_1 is the only vector fixed by all such matrices. □

Proposition 5.5. *Let $T = (T_1, \dots, T_n)$ be a vector of bounded linear transformations on $L^2\mathbb{R}^n$. Suppose each T_i commutes with translation and positive dilation, and hence by fact 4.8 we have functions m_i homogenous of degree 0 such that*

$$\mathcal{F}T_i = m_i\mathcal{F}$$

Then if $m = (m_1, \dots, m_n)$ commutes with rotation, T is a constant multiple of R .

Proof. By lemma 5.4 we know $m(\zeta) = C \frac{\zeta}{|\zeta|}$. It suffices to show the multiplier of R also commutes rotation.

Clearly R satisfies the conditions of theorem 4.16, with $\Omega = Id_{S^{n-1}}$. Therefore the multiplier m^R of R is given by the formula

$$(5.1) \quad m^R(x) = \int_{S^{n-1}} \left(\frac{\pi i}{2} \text{sign}(x \cdot y) - \log|x \cdot y| \right) y dy$$

and hence m^R commutes with rotation. In fact, evaluating (5.1) at a point gives $m^R(\zeta) = i \frac{\zeta}{|\zeta|}$. $\square$

Application 5.6 (L^p estimate for elliptic operators). Suppose $\Delta u = f$ for $f \in C_c^1(\mathbb{R}^n)$. Then $\|u\|_{\dot{W}^{2,p}\mathbb{R}^n} \leq C\|f\|_{L^p\mathbb{R}^n}$ for $1 < p < \infty$.

Proof. It suffices to show

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \right\|_{L^p} \leq C\|f\|_{L^p}$$

Using that $m(\zeta) = |\zeta|^2$ is the multiplier of Δ , we have

$$\begin{aligned} \mathcal{F}(\partial_i \partial_j u)(\zeta) &= -4\pi^2 \zeta_i \zeta_j \mathcal{F}u(\zeta) \\ &= 4\pi^2 \frac{i\zeta_i}{|\zeta|} \frac{i\zeta_j}{|\zeta|} |\zeta|^2 \mathcal{F}u(\zeta) \\ &= -\mathcal{F}(R_i R_j \Delta u)(\zeta) \\ &= -\mathcal{F}(R_i R_j f)(\zeta) \end{aligned}$$

The result follows from the L^p bound of theorem 4.16. $\square$

Application 5.7. If $f \in C_c^1(\mathbb{R}^2)$, then

$$\left\| \frac{\partial f}{\partial x_1} \right\|_{L^p} + \left\| \frac{\partial f}{\partial x_2} \right\|_{L^p} \leq A_p \left\| \frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} \right\|_{L^p}$$

for $1 < p < \infty$

Proof. Follows directly from the relation

$$\frac{\partial f}{\partial x_i} = -R_i(R_1 - iR_2)\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right)$$

and the L^p bounds of theorem 4.16 $\square$

6. POISSON INTEGRAL

Given $f(x) \in L^2(\mathbb{R}^n)$, we want to find $u(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, such that

$$(6.1) \quad \begin{cases} \Delta_{\mathbb{R}_+^{n+1}} u(x, y) &= 0, \\ u(x, 0) &= f(x). \end{cases}$$

Denoting by $\hat{u}(t, y)$ the partial Fourier transform of u in x , we obtain $\hat{u}(t, y) = \hat{f}(t)e^{-2\pi|t|y}$ and thus

$$(6.2) \quad u(x, y) = \int_{\mathbb{R}^n} \hat{f}(t) e^{-2\pi|t|y} e^{-2\pi i x \cdot t} dt.$$

That u satisfies the Poisson equation $\Delta u = 0$ follows from

$$\Delta_{\mathbb{R}_+^{n+1}} (e^{-2\pi|t|y} e^{-2\pi i x \cdot t}) = 0,$$

where $\Delta_{\mathbb{R}_+^{n+1}} = \frac{\partial^2}{\partial y^2} + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Moreover, $u(x, 0) = f(x)$ holds for some classes of f . As a simple example, suppose $f \in L^2(\mathbb{R}^n)$, then $u(x, y) \rightarrow f(x)$ as $y \rightarrow 0$ in $L^2(\mathbb{R}^n)$ -norm; indeed, by Plancherel,

$$\|u(x, y) - f(x)\|_{L^2} = \|\hat{u}(t, y) - \hat{f}(t)\|_{L^2} = \|\hat{f}(t)(e^{-2\pi|t|y} - 1)\|_{L^2} \xrightarrow{y \rightarrow 0} 0$$

by the Dominated Convergence Theorem, since $|e^{-2\pi|t|y} - 1| \leq 2$ for $y \geq 0$, and we have pointwise convergence $e^{-2\pi|t|y} \rightarrow 0$ as $y \rightarrow 0$. This holds more generally:

Theorem 6.1. *In (6.1), suppose that $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then for the solution (6.2) of (6.1), we have $u(x, y) \rightarrow f(x)$ in $L^p(\mathbb{R}^n)$ -norm.*

We will deduce this from the following lemma:

Lemma 6.2. *Let $\phi \in L^1(\mathbb{R}^n)$. Set $\psi(x) = \sup_{|y| \geq |x|} |\phi(y)|$, and suppose $A := \int \psi(x) dx < \infty$. Moreover, let $\phi_\epsilon(x) = \epsilon^{-n} \phi(\epsilon^{-1}x)$. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then:*

- (1) $\sup_{\epsilon > 0} |f * \phi_\epsilon|(x) \leq AMf(x)$, where Mf is the maximal function of f .
- (2) If $\int_{\mathbb{R}^n} \phi(x) dx = 1$, then $\lim_{\epsilon \rightarrow 0} (f * \phi_\epsilon)(x) = f(x)$ a.e. x .
- (3) If $\int_{\mathbb{R}^n} \phi(x) dx = 1$, then $\|f * \phi_\epsilon - f\|_{L^p} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. We only prove the last part. One very easy way to proceed is to first show the stated convergence for continuous, compactly supported f and then use a density argument. Alternatively, observe that

$$\|f * \phi_\epsilon - f\|_{L^p} = \left\| \int (f(x - y) - f(x)) \phi_\epsilon(y) dy \right\|_{L^p} \leq \int \|f(\cdot - y) - f\|_{L^p} |\phi_\epsilon(y)| dy.$$

Now, given $\delta > 0$, we will show that we can choose $r > 0$ such that for $|y| \leq r$, $\|f(\cdot - y) - f\|_{L^p} < \delta$. But then

$$\begin{aligned} \|f * \phi_\epsilon - f\|_{L^p} &\leq \int_{|y| \leq r} \|f(\cdot - y) - f\|_{L^p} |\phi_\epsilon(y)| dy + \int_{|y| > r} \|f(\cdot - y) - f\|_{L^p} |\phi_\epsilon(y)| dy \\ &\leq \delta \|\phi\|_{L^1} + 2\|f\|_{L^p} \int_{|y| > r\epsilon^{-1}} |\phi(y)| dy < \delta(1 + \|\phi\|_{L^1}) \end{aligned}$$

for sufficiently small $\epsilon > 0$. Since $\delta > 0$ was arbitrary, this proves the lemma.

To finish the proof, we show that for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, we have $\Delta(y) := \|f(\cdot + y) - f\|_{L^p} \rightarrow 0$ as $y \rightarrow 0$. This is clearly true for continuous, compactly supported f . For general $f \in L^p(\mathbb{R}^n)$, decompose $f = f_1 + f_2$, where f_1 is continuous with compact support, $f_2 \in L^p(\mathbb{R}^n)$, and $\|f_2\|_{L^p} < \delta$; then

$$\Delta(y) \leq \|f_1(\cdot + y) - f_1\|_{L^p} + 2\delta;$$

but the first summand on the right converges to 0 as $y \rightarrow 0$, and we are done. $\square$

Remark 6.3. The last part of the lemma is false for $p = \infty$: Indeed, taking f to be the characteristic function of an interval and ϕ with compact support, it is easy to see that $\|f * \phi_\epsilon - f\|_{L^\infty} = 1/2$ for all sufficiently small $\epsilon > 0$.

On the other hand, part (2) is true for $p = \infty$. Indeed, it suffices to show that if $f \in L^\infty(\mathbb{R}^n)$, then

$$\lim_{\epsilon \rightarrow 0} (f * \phi_\epsilon)(x) = f(x)$$

a.e. $x \in B$, for every fixed compact ball B . To see this, let $B_1 \supset B$ be a strictly bigger ball, and write $f = f_1 + f_2$ with

$$f_1(x) = \begin{cases} f(x), & x \in B_1 \\ 0, & x \notin B_1. \end{cases}$$

Then $f_1 \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$, so the statement holds for f_1 . For f_2 , we have

$$(f_2 * \phi_\epsilon)(x) = \int f_2(y) \phi_\epsilon(x - y) dy,$$

and in the support of the integrand, $y \in B_1$, thus for $x \in B$, we have $|x - y| \geq \delta = \text{dist}(B, B_1)$; so for $x \in B$,

$$|(f_2 * \phi_\epsilon)(x)| \leq \|f_2\|_{L^\infty} \int_{B_1} |\phi_\epsilon(x - y)| dy \leq \|f_2\|_{L^\infty} \int_{|y| > \delta} |\phi_\epsilon(y)| dy \xrightarrow{\epsilon \rightarrow 0} 0.$$

We can now prove Theorem 6.1.

Proof of Theorem 6.1. By (6.2), we have $u(x, y) = (P_y * f)(x)$, where $\widehat{P_y}(t) = e^{-2\pi|t|y}$. Computing the inverse Fourier transform gives

$$P_y(x) = \frac{c_n y}{(|x|^2 + |y|^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}.$$

Notice that P_y is homogeneous of degree $-n$ with respect y , that is, $P_y(x) = y^{-n} P_1(y^{-1}x)$. Moreover, $P_1(x)$ is positive, decreasing in $|x|$, in L^p for $1 \leq p \leq \infty$, and $\int_{\mathbb{R}^n} P_y(x) dx = \widehat{P_y}(0) = 1$. Therefore, we can apply Lemma 6.2 with $\phi = \psi = P_1$. $\square$

7. SPHERICAL HARMONICS

Definition 7.1. Let $\mathcal{P}_k$ be the linear space of homogeneous polynomials of degree k with complex coefficients in $\mathbb{R}^n$. Let $\mathcal{H}_k$ be the subspace of harmonic homogeneous degree k polynomials with complex coefficients.

We have natural orthogonality properties between $\mathcal{H}_j$ and $\mathcal{H}_k$ for $j \neq k$: Indeed, if $P(x) \in \mathcal{H}_j, Q(x) \in \mathcal{H}_k$, then

$$\begin{aligned} (j-k) \int_{\mathbb{S}^{n-1}} P(x) \overline{Q(x)} d\sigma(x) &= \int_{\mathbb{S}^{n-1}} \overline{Q(x)} \frac{\partial P}{\partial n}(x) - P(x) \overline{\frac{\partial Q}{\partial n}(x)} d\sigma(x) \\ &= \int_{|x| \leq 1} (\overline{Q} \Delta P - P \overline{\Delta Q}) dx = 0, \end{aligned}$$

thus

$$\int_{\mathbb{S}^{n-1}} P(x) \overline{Q(x)} d\sigma(x) = 0.$$

Lemma 7.2. Every $P \in \mathcal{P}_k$ can be uniquely written as $P = P_1 + |x|^2 P_2$, where $P_1 \in \mathcal{H}_k$ and $P_2 \in \mathcal{P}_{k-2}$.

Proof. Write $P(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $x^\alpha = \prod x_j^{\alpha_j}$, $|\alpha| = \sum \alpha_j$. Define

$$P \left(\frac{\partial}{\partial x} \right) = \sum_{|\alpha|=k} a_\alpha \left(\frac{\partial}{\partial x} \right)^\alpha,$$

where $\left(\frac{\partial}{\partial x} \right)^\alpha = \frac{\partial_1^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial_n^{\alpha_n}}{\partial x_n^{\alpha_n}}$. Define an inner product on $\mathcal{P}_k$ by

$$\langle P, Q \rangle = P \left(\frac{\partial}{\partial x} \right) \overline{Q}.$$

To see that this is indeed an inner product, notice that if P and Q are different monomials, then $\langle P, Q \rangle = \langle Q, P \rangle = 0$ in view of $\left(\frac{\partial}{\partial x} \right)^\alpha x^\beta = 0$ if $\alpha \neq \beta$, and thus

$$\langle P, P \rangle = \sum_{|\alpha|=k} |a_\alpha|^2 \alpha!,$$

where $\alpha! = \prod \alpha_j!$. We now claim that $|x|^2 \mathcal{P}_{k-2}$ is the orthogonal complement of $\mathcal{H}_k$ with respect to this inner product, which would finish the proof. To show the inclusion $\mathcal{H}_k \in (|x|^2 \mathcal{P}_{k-2})^\perp$, observe that for all $Q \in \mathcal{H}_k, P \in \mathcal{P}_{k-2}$, we have

$$\langle |x|^2 P, Q \rangle = \langle P, \Delta Q \rangle = 0.$$

For the converse inclusion, suppose $P_1 \in (|x|^2 \mathcal{P}_{k-2})^\perp$, then

$$\langle |x|^2 P_2, P_1 \rangle = 0 \quad \forall P_2 \in \mathcal{P}_{k-2},$$

so $\langle P_2, \Delta P_1 \rangle = 0$ for all $P_2 \in \mathcal{P}_{k-2}$. Choosing $P_2 = \Delta P_1$ gives $\langle \Delta P_1, \Delta P_1 \rangle = 0$, and since $\langle \cdot, \cdot \rangle$ is an inner product, $\Delta P_1 = 0$, as was to be shown. $\square$

Iterating this, we can decompose any polynomial $P \in \mathcal{P}_k$ as a finite sum $P = P_1 + |x|^2 P_2 + |x|^4 P_3 + \cdots$ with $P_j \in \mathcal{H}_{k-2(j-1)}$; restricting to $\mathbb{S}^{n-1}$, this gives $P = P_1 + P_2 + P_3 + \cdots$.

Definition 7.3. Define H_k to be the linear space of restrictions of function in $\mathcal{H}_k$ to $\mathbb{S}^{n-1}$.

Relative to the standard inner product on $L^2(\mathbb{S}^{n-1})$, we have $H_j \perp H_k$ for $j \neq k$. Moreover, $\sum_{k=0}^{\infty} H_k$ is dense in $L^2(\mathbb{S}^{n-1})$; indeed, polynomials in $\mathbb{R}^n$ are dense in $C(\overline{B_1})$ by Stone-Weierstraß, thus the space of restrictions of polynomials to $\mathbb{S}^{n-1}$ is dense in $C(\mathbb{S}^{n-1})$, which in turn is dense in $L^2(\mathbb{S}^{n-1})$; but every polynomial is the sum of homogeneous polynomials, which, whose restrictions to $\mathbb{S}^{n-1}$ are in turn sums of restrictions of harmonic polynomials to $\mathbb{S}^{n-1}$. In summary:

Theorem 7.4. *The inclusions $H_k \hookrightarrow L^2(\mathbb{S}^{n-1})$, $k = 0, 1, 2, \dots$, induce an isometric isomorphism $L^2(\mathbb{S}^{n-1}) \cong \bigoplus_{k=0}^{\infty} H_k$.*

Thus, for all $f \in L^2(\mathbb{S}^{n-1})$, we can find $\{Y_k\}_{k=0}^{\infty}$ such that

$$f = \sum_{k=0}^{\infty} Y_k = \sum_{k=0}^{\infty} a_k Y_k^0,$$

with convergence in $L^2(\mathbb{S}^{n-1})$, where $Y_k, Y_k^0 \in H_k$, $\|Y_k^0\|_{L^2(\mathbb{S}^{n-1})} = 1$, and

$$\|f\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=0}^{\infty} \int_{\mathbb{S}^{n-1}} |Y_k|^2 d\sigma = \sum_{k=0}^{\infty} |a_k|^2.$$

Also note that since H_0 consists of constant functions, $\int_{\mathbb{S}^{n-1}} Y_k d\sigma = 0$ for $k \neq 0$ by the orthogonality of H_0 and H_k ; thus

$$Y_0 = \frac{1}{\text{vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x) d\sigma(x).$$

The Y_k and Y_k^0 are eigenfunctions of $\Delta_{\mathbb{S}^{n-1}}$; more precisely:

Proposition 7.5. *If $Y_k \in H_k$, then $\Delta_{\mathbb{S}^{n-1}} Y_k = -k(k+n-2)Y_k$.*

Proof. Since

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}},$$

we see that $\Delta_{\mathbb{S}^{n-1}} Y_k$ equals the restriction to $\mathbb{S}^{n-1}$ of $\Delta_{\mathbb{R}^n}$ acting on the homogeneous degree 0 extension of Y_k , which is equal to $|x|^{-k} P_k(x)$ for some $P_k \in \mathcal{H}_k$. Therefore,

$$\Delta_{\mathbb{S}^{n-1}} Y_k = (\Delta_{\mathbb{R}^n} P_k)|x|^{-k} + P_k(\Delta_{\mathbb{R}^n} |x|^{-k}) + 2 \sum_{j=1}^n \frac{\partial}{\partial x_j} (|x|^{-k}) \frac{\partial}{\partial x_j} P_k(x).$$

The first summand vanishes since P_k is harmonic on $\mathbb{R}^n$; for the second summand, we compute

$$\Delta_{\mathbb{R}^n} |x|^{-k} = (k(k+1) - (n-1)k)|x|^{-k-2} = k(k-n+2)|x|^{-k-2},$$

and for the third summand, we compute $\frac{\partial}{\partial x_j}|x|^{-k} = -kx_j|x|^{-k-2}$ and

$$\sum_{j=1}^n x_j \frac{\partial}{\partial x_j} P_k(x) = kP_k(x)$$

by homogeneity. Thus, at $|x| = 1$, using $P_k|_{\mathbb{S}^{n-1}} = Y_k$,

$$\Delta_{\mathbb{S}^{n-1}} Y_k = (k(k-n+2) - 2k^2)Y_k = -k(k+n-2)Y_k. \quad \square$$

The expansion of $f = \sum_{k=0}^{\infty} a_k Y_k^0$, $\|Y_k^0\|_{L^2} = 1$, as above, into spherical harmonics is very similar to the Fourier decomposition of functions on the circle. As an example, we have:

Lemma 7.6. *The function $f = \sum_{k=0}^{\infty} a_k Y_k^0$ is C^∞ if and only if for all N , there exists a constant C_N such that $|a_k| \leq C_N k^{-N}$.*

Proof. If $f \in C^\infty$, then for all $r \in \mathbb{N}$, we can integrate by parts to obtain

$$\int (\Delta^r f) Y_k^0 d\sigma = a_k (-k(k+n-2))^r.$$

But by Cauchy-Schwarz, the left hand side is uniformly bounded in k , hence $a_k = \mathcal{O}(k^{-2r})$ for all $r \in \mathbb{N}$. Conversely, $a_k = \mathcal{O}(k^{-N})$ for all N implies $\Delta^r f \in L^2$ for all $r \in \mathbb{N}$. By elliptic regularity, this implies $f \in C^\infty$. $\square$

Next, we use spherical harmonics to generalize the Riesz transform.

Lemma 7.7. *Let $P_k \in \mathcal{H}_k$. Then*

$$\mathcal{F}(P_k(x)e^{-\pi|x|^2}) = i^k P_k(x)e^{-\pi|x|^2}.$$

Proof. For fixed $t \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2} P_k(x+t) dx = \int_0^\infty r^{n-1} e^{-\pi r^2} \int_{\mathbb{S}^{n-1}} P_k(t+r\omega) d\omega dr.$$

Since P_k is harmonic, the inner integral equals $\omega_{n-1} P_k(t)$, where ω_{n-1} is the area of $\mathbb{S}^{n-1}$, hence

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2} P_k(x+t) dx = P_k(t) \int_{\mathbb{R}^n} e^{-\pi|x|^2} dx = P_k(t).$$

Both sides of this equation are entire in t , hence for $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2} P_k(x-iy) dx = P_k(-iy) = (-i)^k P_k(y).$$

Changing variables to $x-iy$ in the integral and using Cauchy's Theorem to shift the contour of integration back to $\mathbb{R}^n$ again gives

$$(-i)^k P_k(y) = \int_{\mathbb{R}^n} e^{-\pi|x|^2 + \pi|y|^2 - 2\pi i x \cdot y} P_k(x) dx,$$

which upon multiplication by $e^{-\pi|y|^2}$ and changing the integration variable to $-x$, thereby picking up an additional factor of $(-1)^k$ from the homogeneity of P_k , proves the result. $\square$

Corollary 7.8. *Fix a nonzero $P_k \in \mathcal{H}_k$. For radial functions $f(x) = f(|x|)$ with $P_k(x)f(r) \in L^2(\mathbb{R}^n)$, the Fourier transform of $P_k(x)f(r)$ is also of the form $P_k(x)g(r)$ with g radial, and the map $\mathcal{F}_{n,k}$ defined by $\mathcal{F}_{n,k}(f) = g$ essentially only depends on $n + 2k$ (but not on P_k), more precisely,*

$$\mathcal{F}_{n,k} = i^k \mathcal{F}_{n+2k,0}.$$

Proof. By the lemma, we know that if $f = e^{-\pi|x|^2}$, then the statement is true. Next, if $f(r) = e^{-\pi\delta r^2}$ with $\delta > 0$, we compute

$$\begin{aligned} \mathcal{F}(P_k(x)e^{-\pi\delta r^2}) &= \delta^{-k/2} \mathcal{F}(P_k(\delta^{1/2}x)e^{-\pi\delta r^2}) = \delta^{-k/2} \delta^{-n/2} \mathcal{F}(P_k(\cdot)e^{-\pi|\cdot|^2}) \left(\frac{x}{\delta^{1/2}} \right) \\ &= \delta^{-k/2} \delta^{-n/2} i^k P_k \left(\frac{x}{\delta^{1/2}} \right) e^{-\pi x^2/\delta} = \delta^{-k} \delta^{-n/2} i^k P_k(x) e^{-\pi x^2/\delta}, \end{aligned}$$

thus

$$\mathcal{F}_{n,k}(e^{-\pi\delta r^2}) = i^k \delta^{-k-n/2} e^{-\pi r^2/\delta},$$

which implies

$$\mathcal{F}_{n,k}(e^{-\pi\delta r^2}) = i^k \mathcal{F}_{n+2k,0}(e^{-\pi\delta r^2}).$$

This implies the lemma for all f which are in the closure of the span of $\{e^{-\pi\delta r^2}\}_{\delta>0}$ in the Hilbert space

$$L^2((0, \infty), r^{2k+n-1} dr) = \left\{ f(r) \mid \int_0^\infty |f(r)|^2 r^{2k+n-1} dr < \infty \right\},$$

which is precisely the space of all radial functions f for which $P_k(x)f(r) \in L^2(\mathbb{R}^n)$. But it is in fact easy to see that the span of $\{e^{-\pi\delta r^2}\}_{\delta>0}$ is dense in $L^2((0, \infty), r^{2k+n-1} dr)$: Indeed, if $f \in L^2((0, \infty), r^{2k+n-1} dr)$ is such that

$$\int_0^\infty f(r) e^{-\pi\delta r^2} r^{2k+n-1} dr = 0 \quad \forall \delta > 0,$$

then a change of variables shows that the Laplace transform of $f(\sqrt{r})r^{(2k+n-2)/2} \in L^1((0, \infty), dr)$ is 0, thus $f \equiv 0$. $\square$

We now generalize the Riesz transform:

Theorem 7.9. *Let $P_k \in \mathcal{H}_k(\mathbb{R}^n)$, $k \geq 1$. Then the multiplier of the kernel $|x|^{-n-k}P_k(x)$ (in the sense of taking the principal value) is $\gamma_k|x|^{-k}P_k$, where*

$$\gamma_k = \frac{i^k \pi^{n/2} \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)}.$$

We will use an approximation argument.

Lemma 7.10. *For $0 < \alpha < n$, $k \geq 0$, we have*

$$(7.1) \quad \mathcal{F} \left(\frac{P_k(x)}{|x|^{n+k-\alpha}} \right) = \gamma_{k,\alpha} \frac{P_k(x)}{|x|^{k+\alpha}},$$

in the sense that

$$\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{n+k-\alpha}} \hat{\phi}(x) dx = \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}} \phi(x) dx$$

for $\phi \in C_c^\infty(\mathbb{R}^n)$. Here

$$\gamma_{k,\alpha} = \frac{i^k \pi^{n/2-\alpha} \Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k+n-\alpha}{2}\right)}.$$

Proof. Since $\mathcal{F}(P_k(x)e^{-\pi\delta|x|^2}) = i^k \delta^{-k-n/2} e^{-\pi|x|^2/\delta} P_k(x)$, we have

$$\int_{\mathbb{R}^n} P_k(x) e^{-\pi\delta|x|^2} \hat{\phi}(x) dx = i^k \delta^{-k-n/2} \int_{\mathbb{R}^n} P_k(x) e^{-\pi|x|^2/\delta} \phi(x) dx.$$

If we multiply both sides by $\delta^{(k+n-\alpha)/2-1}$ and integrate with respect to δ from 0 to ∞ , the left hand side becomes

$$\int \frac{P_k(x) \hat{\phi}(x)}{|x|^{k+n-\alpha}} dx \cdot \Gamma\left(\frac{k+n-\alpha}{2}\right) \pi^{-(k+n-\alpha)/2},$$

where we used $\int_0^\infty e^{-\pi\delta|x|^2} \delta^{\beta-1} d\delta = (\pi|x|^2)^{-\beta} \Gamma(\beta)$, and the right hand side becomes

$$i^k \Gamma\left(\frac{k+\alpha}{2}\right) \pi^{-(k+\alpha)/2} \int_{\mathbb{R}^n} \frac{P_k(x) \phi(x)}{|x|^{k+\alpha}} dx. \quad \square$$

Proof of Theorem 7.9. We want to take $\alpha \rightarrow 0$ in the previous Lemma. For $\phi \in C_c^\infty(\mathbb{R}^n)$, we can take the limit $\alpha \rightarrow 0$ in the right hand side of (7.1) directly, obtaining

$$\lim_{\alpha \rightarrow 0+} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}} \phi(x) dx = \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^k} \phi(x) dx.$$

For the left hand side, we split the integral, writing

$$\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{n+k-\alpha}} \hat{\phi}(x) dx = \int_{|x| \leq 1} \frac{P_k(x)}{|x|^{n+k-\alpha}} \hat{\phi}(x) dx + \int_{|x| > 1} \frac{P_k(x)}{|x|^{n+k-\alpha}} \hat{\phi}(x) dx.$$

In the second integral, we can take the limit $\alpha \rightarrow 0+$ directly. To deal with the first term, we use the cancellation property

$$\int_{r < |x| < R} \frac{P_k(x)}{|x|^{n+k-\alpha}} dx = 0$$

for all $0 < r < R$, which follows from the fact that the integral of P_k over any coordinate sphere vanishes in view of $k \geq 1$, to rewrite the first integral as

$$\int_{|x| \leq 1} \frac{P_k(x)}{|x|^{n+k-\alpha}} \left(\hat{\phi}(x) - \hat{\phi}(0) \right) dx \xrightarrow{\alpha \rightarrow 0} \int_{|x| \leq 1} \frac{P_k(x)}{|x|^{n+k}} \left(\hat{\phi}(x) - \hat{\phi}(0) \right) dx,$$

where the integrand is now integrable, since $\hat{\phi}(x) - \hat{\phi}(0) = \mathcal{O}(x)$. The last integral in turn can be rewritten as

$$\begin{aligned} \int_{|x| \leq 1} \frac{P_k(x)}{|x|^{n+k}} \left(\hat{\phi}(x) - \hat{\phi}(0) \right) dx &= \lim_{\epsilon \rightarrow 0+} \int_{\epsilon \leq |x| \leq 1} \frac{P_k(x)}{|x|^{n+k}} \left(\hat{\phi}(x) - \hat{\phi}(0) \right) dx \\ &= \lim_{\epsilon \rightarrow 0+} \int_{\epsilon \leq |x| \leq 1} \frac{P_k(x)}{|x|^{n+k}} \hat{\phi}(x) dx. \end{aligned}$$

Thus,

$$\lim_{\epsilon \rightarrow 0+} \int_{|x| \geq \epsilon} \frac{P_k(x)}{|x|^{n+k}} \hat{\phi}(x) dx = \gamma_k \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^k} \phi(x) dx,$$

where $\gamma_k = \gamma_{k,0}$. Now, taking $\hat{\phi}(x) = f(y-x)$, thus $\phi(x) = \hat{f}(x)e^{-2\pi i x \cdot y}$, we obtain

$$Tf(y) := \lim_{\epsilon \rightarrow 0+} \int_{|x| \geq \epsilon} \frac{P_k(x)}{|x|^{n+k}} f(y-x) dx = \gamma_k \int \frac{P_k(x)}{|x|^k} \hat{f}(x) e^{-2\pi i x \cdot y} dx;$$

but the multiplier of T is, by definition, the function m with $\widehat{Tf} = m\hat{f}$, i.e.

$$Tf(y) = \int m(x) \hat{f}(x) e^{-2\pi i x \cdot y} dx,$$

and we can therefore read off $m(x) = \gamma_k \frac{P_k(x)}{|x|^k}$, finishing the proof. $\square$

Theorem 7.11. *The following two classes of transforms on $L^2(\mathbb{R}^n)$ are identical:*

- (1) $Tf = cf + \lim_{\epsilon \rightarrow 0} \int_{|y| \geq \epsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy$, where Ω is homogeneous of degree 0, $\int_{\mathbb{S}^{n-1}} \Omega d\sigma = 0$, and $\Omega \in C^\infty(\mathbb{S}^{n-1})$,
- (2) $\widehat{Tf} = m\hat{f}$, where m is homogeneous of degree 0, and $m \in C^\infty(\mathbb{S}^{n-1})$.

Proof. Use $\Omega(x) = \sum_{k=1}^{\infty} Y_k$, with the Y_0 -term absent because of $\int_{\mathbb{S}^{n-1}} \Omega d\sigma = 0$, and $m(x) = \sum_{k=0}^{\infty} \tilde{Y}_k$, and the previous theorem, which in fact gives $\tilde{Y}_k = \gamma_k Y_k$. $\square$

As an application of 7.11, let's consider linear elliptic operators. Suppose P is a homogeneous degree k polynomial. P is said to be **elliptic** if $p(x) \neq 0$, $\forall x \neq 0$.

Theorem 7.12. *Let P be as above and assume that $f \in C_c^k$. Then*

$$\left\| \left(\frac{\partial}{\partial x} \right)^\alpha f \right\|_{L^p} \leq C_p \left\| P \left(\frac{\partial}{\partial x} \right) f \right\|_{L^p}, \quad \forall |\alpha| = k, 1 < p < \infty$$

Proof. Given $|\alpha| = k$, define $m(y) = \frac{y^\alpha}{p(y)}$ and let T be the operator with multiplier m . Then T has the nice property that

$$\left(\frac{\partial}{\partial x} \right)^\alpha f = T \left(P \left(\frac{\partial}{\partial x} \right) f \right)$$

Now since m is smooth on $\mathbb{S}^{n-1}$ and homogeneous of degree 0, by 7.11 and the theory of singular integrals developed earlier, we have

$$\left\| \left(\frac{\partial}{\partial x} \right)^\alpha f \right\|_{L^p} \leq C_p \left\| P \left(\frac{\partial}{\partial x} \right) f \right\|_{L^p}$$

$\square$

8. LITTLEWOOD-PALEY THEORY

Let $f \in L^p$ and let

$$u(x, y) = \int_{\mathbb{R}^n} f(x - t) P_y(t) dt, x \in \mathbb{R}^n, y \in \mathbb{R}_+$$

be it's Poisson integral. We let $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and let ∇ denote the gradient in both x and y .

Definition 8.1 (g -function). For $f \in L^p$, define

$$\begin{aligned} g(f)(x) &= \left(\int_0^\infty |\nabla u(x, y)|^2 y dy \right)^{1/2}, \\ g_1(f)(x) &= \left(\int_0^\infty \left| \frac{\partial}{\partial y} u(x, y) \right|^2 y dy \right)^{1/2}, \\ g_2(f)(x) &= \left(\int_0^\infty |\nabla_x u(x, y)|^2 y dy \right)^{1/2}, \end{aligned}$$

Note that $g_1^2 + g_2^2 = g^2$.

It turns out that the L^p -norm of $g(f)$ is comparable to that of f itself. More precisely, we have the following result:

Theorem 8.2. Let $f \in L^p(\mathbb{R}^n)$ where $p \in (1, \infty)$. Then $g(f) \in L^p(\mathbb{R}^n)$. Moreover,

$$(8.1) \quad \tilde{A}_p \|f\|_{L^p} \leq \|g(f)\|_{L^p} \leq A_p \|f\|_{L^p}$$

Proof. First we notice that the case $p = 2$ follows from Plancherel's identity. Indeed, we can write

$$\begin{aligned} u(x, y) &= \int_{\mathbb{R}^n} \hat{f}(t) e^{-2\pi|t|y} e^{-2\pi i x \cdot t} dt \\ \frac{\partial u}{\partial y}(x, y) &= \int_{\mathbb{R}^n} (-2\pi|t|) \hat{f}(t) e^{-2\pi|t|y} e^{-2\pi i x \cdot t} dt \\ \frac{\partial u}{\partial x_i} &= \int_{\mathbb{R}^n} (-2\pi i t_i) \hat{f}(t) e^{-2\pi|t|y} e^{-2\pi i x \cdot t} dt \end{aligned}$$

So by Plancherel's identity,

$$\int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial y}(x, y) \right|^2 dx = \int_{\mathbb{R}^n} 4\pi^2 |t|^2 |\hat{f}(t)|^2 e^{-4\pi|t|y} dt = \int_{\mathbb{R}^n} |\nabla_x u(x, y)|^2 dx$$

Therefore the L^2 -norm of $g_1(f)$ is given by

$$\begin{aligned} \|g_1(f)\|_{L^2}^2 &= \int_{\mathbb{R}^n} \int_0^\infty \left| \frac{\partial u(x, y)}{\partial y} \right|^2 y dy dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} y 4\pi^2 |t|^2 e^{-4\pi|t|y} |\hat{f}(t)|^2 dt dy \\ &= \int_{\mathbb{R}^n} |\hat{f}(t)|^2 \left[4\pi^2 |t|^2 \int_0^\infty e^{-4\pi|t|y} y dy \right] dt \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} |\hat{f}(t)|^2 \left[\frac{1}{4} \int_0^\infty e^{-z} z dz \right] dt \\
&= \frac{1}{4} \|f\|_{L^2}^2
\end{aligned}$$

Similarly, we can show that

$$\|g_2(f)\|_{L^2}^2 = \frac{1}{4} \|f\|_{L^2}^2$$

Hence, using the relation $g_1^2 + g_2^2 = g^2$, we have

$$\|g(f)\|_{L^2}^2 = \frac{1}{2} \|f\|_{L^2}^2.$$

For general p , we will apply the vector-valued version of singular integral theory developed earlier. Define the following two Hilbert spaces:

$$\mathcal{H}_1 = \mathbb{C}; \quad \mathcal{H}_2 = \{(f_0, f_1, \dots, f_n) | f_i \in H_2^0\}$$

$$H_2^0 = \{f | \int_0^\infty |f|^2 y dy < \infty\}$$

Then since $\mathcal{H}_1 = \mathbb{C}$, the space of bounded linear transforms $B(\mathcal{H}_1, \mathcal{H}_2)$ is isomorphic to $\mathcal{H}_2$ itself.

Next we define the kernel K_ε :

$$K_\varepsilon(x) = \left(\frac{\partial P_{y+\varepsilon}}{\partial y}, \frac{\partial P_{y+\varepsilon}}{\partial x_1}, \dots, \frac{\partial P_{y+\varepsilon}}{\partial x_n} \right)$$

and let $T_\varepsilon f(x) = \int_{\mathbb{R}^n} K_\varepsilon(t) f(x-t) dt$. We want to verify the following properties of K_ε in order to apply singular integral theory:

- (1) $K_\varepsilon(x)$ takes value in $\mathcal{H}_2 = B(\mathcal{H}_1, \mathcal{H}_2)$.
- (2) $K_\varepsilon \in L^2(\mathbb{R}^n, \mathcal{H}_2)$
- (3) $|\frac{\partial K_\varepsilon}{\partial x_i}| \leq A|x|^{-n-1}$
- (4) $|\hat{K}_\varepsilon(t)| \leq 1/\sqrt{2}$

Recalling the definition of the Poisson kernel

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{n+1/2}}$$

So it's not hard to check that

$$\begin{aligned}
\int_0^\infty \left| \frac{\partial P_{y+\varepsilon}}{\partial y} \right|^2 y dy &< \infty \\
\int_0^\infty \left| \frac{\partial P_{y+\varepsilon}}{\partial x_i} \right|^2 y dy &< \infty
\end{aligned}$$

This establishes property (1).

Next we check property (2). Note that

$$|K_\varepsilon(x)|_{\mathcal{H}_2}^2 = \int_0^\infty \left[\left| \frac{\partial P_{y+\varepsilon}}{\partial y} \right|^2 + |\nabla_x P_{y+\varepsilon}|^2 \right] y dy \leq c \int_0^\infty \frac{y dy}{(|x|^2 + |y + \varepsilon|^2)^{n+1}}$$

$$\leq \begin{cases} \int_0^\infty \frac{y dy}{|y + \varepsilon|^{2(n+1)}} = O(\varepsilon^{-2n}) \\ C|x|^{-2n} \end{cases}$$

Therefore $K_\varepsilon \in L^2(\mathbb{R}^n, \mathcal{H}_2)$.

Property (3) can be verified by a direct computation. Finally, notice that

$$|T_\varepsilon f(x)|_{\mathcal{H}_2}^2 = \int_0^\infty |\nabla u(x, y + \varepsilon)|^2 y dy \leq g(f)(x)^2$$

So we have, for each f in $L^2(\mathbb{R}^n)$,

$$\|T_\varepsilon f\|_{L^2(\mathbb{R}, \mathcal{H}_2)} \leq \|g(f)\|_{L^2} = \frac{1}{\sqrt{2}} \|f\|_{L^2}.$$

This gives a L^∞ -bound on the multiplier $\hat{K}_\varepsilon$ and establishes (4), i.e.

$$\|\hat{K}_\varepsilon(t)\| \leq \frac{1}{\sqrt{2}}.$$

Therefore we can apply the vector-valued version of 4.10 and get

$$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n, \mathcal{H}_2)} \leq A_p \|f\|_{L^p}, 1 < p < \infty$$

Now since $\lim_{\varepsilon \rightarrow 0} \nabla u(x, y + \varepsilon) = \nabla u(x, y)$ for all $y > 0$, by Fatou's lemma we have

$$g(f)(x)^2 = \int_0^\infty |\nabla u(x, y)|^2 y dy \leq \liminf_{\varepsilon \rightarrow 0} |T_\varepsilon f(x)|_{\mathcal{H}_2}^2$$

Therefore, using Fatou's lemma again, we get, for $1 < p < \infty$,

$$(8.2) \quad \|g(f)\|_{L^p} \leq \liminf_{\varepsilon \rightarrow 0} \|T_\varepsilon f\|_{L^p(\mathbb{R}, \mathcal{H}_2)} \leq A_p \|f\|_{L^p}$$

Hence we've proved the second inequality in (8.1). Finally we'll use a duality argument to establish the first inequality there.

Recall that

$$\|g(f)\|_{L^2}^2 = \frac{1}{2} \|f\|_{L^2}^2$$

Polarizing this equality gives

$$\frac{1}{2} \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} dx = \int_0^\infty \int_{\mathbb{R}^n} \nabla u_1(x, y) \overline{\nabla u_2(x, y)} y dy dx$$

Hence if we let $q = \frac{p}{p-1}$ be the dual exponent of p and take $f_1 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $f_2 \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $\|f_2\|_{L^q} = 1$, then

$$\begin{aligned} \frac{1}{2} \left| \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} dx \right| &\leq \int_{\mathbb{R}^n} g(f_1) g(f_2) dx \quad (\text{Hölder}) \\ &\leq \|g(f_1)\|_{L^p} \|g(f_2)\|_{L^q} \quad (\text{Hölder again}) \\ &\leq \|g(f_1)\|_{L^p} A_q \|f_2\|_{L^q} \quad (\text{by (8.2)}) \\ &= A_q \|g(f_1)\|_{L^p} \end{aligned}$$

Therefore we get

$$\|f_1\|_{L^p} \leq A_q \|g(f_1)\|_{L^p}$$

This establishes the first inequality of (8.1). $\square$

Remark 8.3. We can define higher-order analogues of the g -function by considering

$$g_k(f) = \left[\int_0^\infty \left| \frac{\partial^k u}{\partial y^k}(x, y) \right|^2 y^{2k-1} dy \right]^{1/2}$$

Then we get $g_{k+1}(f) \geq c_k g_k(f)$ and that $\|g_k(f)\|_{L^p}$ is comparable to $\|f\|_{L^p}$ for $k \geq 1$.

Next we consider another function related to the g -function.

Definition 8.4 (g_λ^* -function). Given $f \in L^p$, let u denote it's Poisson integral, we define

$$(8.3) \quad g_\lambda^*(f)(x) = \left[\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{y}{|t| + y} \right)^{\lambda n} |\nabla u(x - t, y)|^2 y^{1-n} dt dy \right]^{1/2}$$

$$(8.4) \quad S(f)(x) = \left[\int_\Gamma |\nabla u(x - t, y)|^2 y^{1-n} dt dy \right]^{1/2}$$

where Γ is given by

$$\Gamma = \{(t, y) \in \mathbb{R}_+^{n+1} \mid |t| < y\}$$

Theorem 8.5. *We have the following pointwise bounds:*

$$(8.5) \quad g(f)(x) \leq C S(f)(x) \leq C_\lambda g_\lambda^*(f)(x)$$

Proof. We first prove the second inequality in (8.5). This follows from the observation that over Γ ,

$$\left(\frac{y}{|t| + y} \right)^{\lambda n} \geq \left(\frac{1}{2} \right)^{\lambda n}$$

Recalling the definitions (8.3) and (8.4), we get

$$S(f)(x) \leq C_{\lambda, n} g_\lambda^*(f)(x).$$

Next we prove the first inequality in (8.5). Given $y > 0$, we define

$$B_y = \text{The ball centered at } (0, y) \text{ touching } \Gamma$$

Since ∇u is harmonic, by the mean-value property,

$$(8.6) \quad \nabla u(0, y) = \frac{1}{m(B_y)} \int_{B_y} \nabla u(x, t) dx dt \leq \left(\frac{1}{m(B_y)} \int_{B_y} |\nabla u(x, t)|^2 dx dt \right)^{1/2}$$

Therefore we have

$$\begin{aligned} g(f)(0)^2 &= \int_0^\infty y |\nabla u(y, 0)|^2 dy \\ &\leq \int_0^\infty y \frac{1}{m(B_y)} \int_{B_y} |\nabla u(x, t)|^2 dx dt dy \quad (\text{by (8.6)}) \end{aligned}$$

$$\leq \int_0^\infty cy^{-n} \int_{B_y} |\nabla u(x, t)|^2 dx dt dy$$

To continue, note that there exist constants c_1 and c_2 such that

$$(x, t) \in B_y \Rightarrow c_1 t \leq y \leq c_2 t$$

Thus

$$\begin{aligned} g(f)(0)^2 &\leq \int_0^\infty cy^{-n} \int_{B_y} |\nabla u(x, t)|^2 dx dt dy \\ &\leq \int_\Gamma \left[\int_{c_1 t}^{c_2 t} y^{-n} dy \right] |\nabla u(x, t)|^2 dx dt \\ &\leq c \int_\Gamma t^{1-n} |\nabla u(x, t)|^2 dx dt \\ &= cS(f)(0)^2 \end{aligned}$$

Finally note that for all $x \in \mathbb{R}^n$,

$$S(f)(x) = \int_\Gamma |\nabla u(x - t, y)|^2 y^{1-n} dt dy = \int_{\Gamma(x)} |\nabla u(t, y)|^2 y^{1-n} dt dy$$

where $\Gamma(x) = x + \Gamma$ is a shifted cone. Thus if we consider $\Gamma(x)$ instead of Γ in the above argument, we would get

$$g(f)(x) \leq cS(f)(x)$$

this proves the first inequality in (8.5) $\square$

As is the case for the g -function, we can compare the L^p -norm of g_λ^* with that of f . Specifically, we have the following result:

Theorem 8.6. *Let $f \in L^p(1 < p < \infty)$ and suppose that $\lambda > 1$ and $p > 2/\lambda$. Then*

$$(8.7) \quad \|g_\lambda^*\|_{L^p} \leq C_{p,\lambda} \|f\|_{L^p}$$

Proof. We first do the easier case where $p \geq 2$.

Claim 8.7. *For any nonnegative function ψ the following holds:*

$$(8.8) \quad \int_{\mathbb{R}^n} g_\lambda^*(f)(x)^2 \psi(x) dx \leq \int_{\mathbb{R}^n} g(f)(x)^2 M\psi(x) dx$$

Proof of claim. To see this, note that the left-hand side equals

$$\begin{aligned} &\int_{\mathbb{R}^n} \psi(x) \left[\int_{\mathbb{R}^n} \int_0^\infty |\nabla u(x - t, y)|^2 \left(\frac{y}{|t| + y} \right)^{\lambda n} y^{1-n} dt dy \right] dx \\ &= \int_{\mathbb{R}^n} \int_0^\infty y |\nabla u(t, y)|^2 \left[\int_{\mathbb{R}^n} \psi(x) \left(\frac{y}{|t - x| + y} \right)^{\lambda n} y^{-n} dx \right] dt dy \\ &\leq C_{n,\lambda} \int_{\mathbb{R}^n} g(f)(t)^2 M\psi(t) dt \quad (\text{by 1.5 and } \lambda > 1) \end{aligned}$$

This proves the claim. Note that we need $\lambda > 1$ so that $(1+|x|)^{-\lambda n}$ is integrable. $\square$

If we plug $\psi = 1$ into (8.8), then we get (8.7) for $p = 2$. When $2 < p < \infty$, let q be the dual exponent of $p/2$, so that $1 < q < \infty$. Then

$$\begin{aligned} \sup_{\|\psi\|_{L^q}=1} \int_{\mathbb{R}^n} g_\lambda^*(f)(x)^2 \psi(x) dx &\leq \sup_{\|\psi\|_{L^q}=1} \int_{\mathbb{R}^n} g(f)(x)^2 M\psi(x) dx \\ &\leq \|g(f)\|_{L^p}^2 \|M\psi\|_{L^q} \\ &\leq A \|f\|_{L^p}^2 \|\psi\|_{L^q} = A \|f\|_{L^p}^2 \text{ (by 8.1 and 1.6)} \end{aligned}$$

Hence

$$\|g_\lambda^*\|_{L^p}^2 \leq A \|f\|_{L^p}^2$$

and we've established (8.7) for $p \geq 2$, $\lambda > 1$.

To finish the proof in the general case, we will need the following lemmas and definition:

Lemma 8.8. *If u is a harmonic function, then*

$$\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2.$$

Lemma 8.9. *If $F(x, y)$ is continuous on $\overline{\mathbb{R}_+^{n+1}}$ and decays “suitably fast” at ∞ , and is C^2 on $\mathbb{R}_+^{n+1}$, then*

$$\int_{\mathbb{R}_+^{n+1}} y \Delta F(x, y) dx dy = \int_{\mathbb{R}^n} F(x, 0) dx.$$

For example, the decay conditions $|F(x, y)| = O((|x| + |y|)^{-n-\epsilon})$ and $|\nabla F(x, y)| = O((|x| + |y|)^{-n-1-\epsilon})$ are “suitably fast.”

Definition 8.10. We define the weighted maximal function

$$M_\mu(f)(x) = \sup_{r>0} \left(\frac{1}{|B(x, r)|} \int_{B(x, r)} |f|^\mu(y) dy \right)^{\frac{1}{\mu}}$$

Lemma 8.11. *For $f \in L^p(\mathbb{R}^n)$, $p \geq \mu \geq 1$, let $u(x, y)$ denote the Poisson integral of f . Then*

$$\begin{aligned} (1) \quad &|u(x-t, y)| \leq A \left(1 + \frac{|t|}{y}\right)^n M(f)(x) \\ (2) \quad &|u(x-t, y)| \leq A_\mu \left(1 + \frac{|t|}{y}\right)^{\frac{n}{\mu}} M_\mu(f)(x) \end{aligned}$$

Proof. As the statement is invariant under scaling $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda t)$, it suffices to prove the lemma for $y = 1$. Note that

$$|u(x-t, 1)| = |f * P_1(x-t)| \leq A_t M(f)(x),$$

where A_t is the L^1 -integral of the “dominant function” of the Poisson kernel, i.e..

$$\psi_t(x) := c_n \sup_{|x'| \geq |x|} \frac{1}{(1 + |x' - t|)^{\frac{n+1}{2}}}.$$

Because $|\psi_t| \leq C$ for $|x| \leq 2|t|$ and $|\psi_t| \leq A(1 + |x|^2)^{-\frac{n+1}{2}}$ for $|x| > 2|t|$, we see that

$$A_t = \int_{x \in \mathbb{R}^n} \psi_t(x) dx \leq O((1 + |t|)^n).$$

This proves (1). For (2), Hölder's inequality (and $\int_{\mathbb{R}^n} P_y(s) ds = 1$) implies that

$$|u(x - t, y)|^\mu = \left(\int_{s \in \mathbb{R}^n} P_y(s) f(x - t - s) ds \right)^\mu \leq \int_{s \in \mathbb{R}^n} |f|^\mu(x - t - s) P_y(s) ds.$$

We will treat the right hand side as the Poisson integral of $|f|^\mu$ and call it $U(x, y)$. By (1) applied to $U(x, y)$, we obtain

$$\begin{aligned} |u(x - t, y)| &\leq |U(x - t, y)|^{\frac{1}{\mu}} \\ &\leq A^{\frac{1}{\mu}} \left(1 + \frac{|t|}{y} \right)^{\frac{n}{\mu}} M(|f|^\mu)(x)^{\frac{1}{\mu}} \\ &= A_\mu \left(1 + \frac{|t|}{y} \right)^{\frac{n}{\mu}} M_\mu(f)(x). \end{aligned} \quad \square$$

Now, we can finish the proof of the theorem. For $1 < p < 2$ and $\lambda > \frac{2}{p}$, choose $1 \leq \mu < p$ so that

$$\lambda' := \lambda - \frac{2-p}{\mu} > 1.$$

By Lemma 8.11,

$$|u(x - t, y)| \leq C \left(1 + \frac{|t|}{y} \right)^{\frac{n}{\mu}} M_\mu(f)(x).$$

Using this and Lemmas 8.8 and 8.9, we obtain

$$\begin{aligned} (g_\lambda^*(f))^2(x) &= \int_0^\infty \int_{t \in \mathbb{R}^n} y^{1-n} \left(\frac{y}{y + |t|} \right)^{\lambda n} |\nabla u(x - t, y)|^2 dt dy \\ &= \frac{1}{p(p-1)} \int_{\mathbb{R}_+^{n+1}} y^{1-n} \left(\frac{y}{y + |t|} \right)^{\lambda n} u^{2-p}(x - t, y) \Delta(u^p)(x - t, y) dt dy \\ &\leq A_{\mu,p} M_\mu(f)(x)^{2-p} \underbrace{\int_{(t,y) \in \mathbb{R}_+^{n+1}} y^{1-n} \left(\frac{y}{y + |t|} \right)^{\lambda' n} \Delta(u^p)(x - t, y) dt dy}_{:= I^*(x)}. \end{aligned}$$

Note that

$$\begin{aligned} \int_{x \in \mathbb{R}^n} I^*(x) dx &= \int_{\mathbb{R}_+^{n+1}} \int_{x \in \mathbb{R}^n} y^{1-n} \left(\frac{y}{y + |t|} \right)^{\lambda' n} \Delta(u^p)(t, y) dx dt dy \\ &= C_{\lambda',n} \int_{\mathbb{R}^{n+1}} y \Delta(u^p)(t, y) dt dy \\ &= C_{\lambda',n} u^p(t, 0) dt \\ &= C_{\lambda',n} \|f\|_{L^p}^p \end{aligned}$$

The first equality follows from a shift in x and the second follows from the fact that

$$\int_{x \in \mathbb{R}^n} y^{-n} \left(\frac{y}{y + |t - x|} \right)^{\lambda' n} dx = C_{\lambda', n}$$

by a change of variables. Using this, we have

$$\begin{aligned} \int_{x \in \mathbb{R}^n} |g_\lambda^*(f)|^p(x) dx &\leq \int_{x \in \mathbb{R}^n} A_{\mu, p}^{\frac{2}{p}} M_\mu(f)^{\frac{2-p}{2}p}(x) I^*(x)^{\frac{p}{2}} dx \\ &\leq A_{\mu, p} \left(\int_{x \in \mathbb{R}^n} M_\mu(f)(x)^p \right)^{\frac{2-p}{2}} dx \left(\int_{x \in \mathbb{R}^n} I^*(x) dx \right)^{\frac{p}{2}} \\ &\leq A_{\mu, p} \|f\|_{L^p}^{\frac{2-p}{2}p} \|f\|_{L^p}^{\frac{p}{2}} \\ &= A_{\mu, p} \|f\|_{L^p}^p, \end{aligned}$$

which finishes the proof. $\square$

Theorem 8.12. *If $m(x)$ is C^k on $\mathbb{R}^n \setminus \{0\}$ for some $k > \frac{n}{2}$. Assume that for every multi-index α , we have that*

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha m(x) \right| \leq B|x|^{-\alpha}.$$

Then, the associated operator satisfies $\|T_m f\|_{L^p} \leq A_p \|f\|_{L^p}$ for all $f \in L^2 \cap L^p$.

Example 8.13. *If $m(x)$ is homogeneous of degree 0 and C^∞ on the sphere, then it defines a bounded operator $T_m : L^p \rightarrow L^p$.*

Proposition 8.14. *Under the same assumptions on $m(x)$ as in Theorem 8.12, if $f \in L^2(\mathbb{R}^n)$ and $F(x) := T_m f$, then*

$$g_1(F)(x) \leq B_\lambda g_\lambda^*(f)(x),$$

where $\lambda = \frac{2n}{k}$.

This proposition implies the theorem, because if $p \geq 2$, then $\lambda > 1$.

$$\|T_m f\|_{L^p} \leq \|g_1(F)\|_{L^p} \leq B_\lambda \|g_\lambda^*(f)\|_{L^p}$$

(recall that $g_1(f) = \left(\int_0^\infty \left| \frac{\partial}{\partial y} u(x, y) \right|^2 y dy \right)^{\frac{1}{2}}$). If $1 < p < 2$, then we might not have $p > \frac{2}{\lambda}$. However, we can reduce this case to the previous one by duality. Now, we prove the proposition.

Proof. We set

$$\begin{aligned} \hat{u}(x, y) &:= e^{-2\pi|x|y} \hat{f}(x) \\ \hat{U}(x, y) &:= e^{-2\pi|x|y} m(x) \hat{f}(x). \end{aligned}$$

Here, both Fourier transforms are in the x variable only. Let

$$M(x, y) := \int e^{-2\pi i x \cdot t} e^{-2\pi|t|y} m(t) dt.$$

Then,

$$\hat{U}(x, y_1 + y_2) = \hat{M}(x, y_1)\hat{u}(x, y_2),$$

for $y = y_1 + y_2$. In particular,

$$U(x, y_1 + y_2) = \int_{t \in \mathbb{R}^n} M(t, y_1)u(x - t, y_2)dt.$$

Differentiating with respect to y_1 , k -times, and y_2 once, we have that

$$U^{(k+1)}(x, y) = \int_{t \in \mathbb{R}^n} M^{(k)}(t, y_1)u^1(x - t, y_2)dt.$$

Letting $y_1 = y_2 = \frac{y}{2}$, we have

$$U^{(k+1)}(x, y) = \int_{t \in \mathbb{R}^n} M^{(k)}(t, \frac{y}{2})u^1(x - t, \frac{y}{2})dt.$$

The definition of M and decay properties of $m(x)$ imply that

$$\begin{aligned} |M^{(k)}(t, y)| &\leq B'|y|^{-n-k} \\ \|t^\alpha M^{(k)}(t, y)\|_{L^2}^2 &\leq B''y^{-n} \end{aligned}$$

for $|\alpha| \leq k$. Using this,

$$\begin{aligned} |U^{(k+1)}(x, y)| &\leq A|y|^{-n-2k} \underbrace{\int_{|t| \leq \frac{y}{2}} |u^{(1)}(x - t, \frac{y}{2})|^2 dt}_{:= I_1(y)} + A|y|^{-n} \underbrace{\int_{|t| > \frac{y}{2}} \frac{|u^{(1)}(x - t, \frac{y}{2})|^2}{|t|^{2k}} dt}_{:= I_2(y)}, \end{aligned}$$

which implies that

$$\begin{aligned} (g_1(F)(x))^2 &= (g_{k+1}(F)(x))^2 \\ &= \int_0^\infty U^{(k+1)}(x, y)^2 y^{2k+1} dy \\ &\leq \int_0^\infty I_1(y) y^{2k+1} dy + \int_0^\infty I_2(y) y^{2k+1} dy \\ &\leq B \int |\nabla u(x - t, y)|^2 y^{1-n} dt dy + B''(g_\lambda^*(f))^2 \\ &\leq BS(f)(x)^2 + B''(g_\lambda^*(f))^2 \\ &\leq \tilde{B}(g_\lambda^*(f))^2. \end{aligned}$$

The middle inequality follows from the form of g_λ^* and $I_2(y)$. This completes the proof. $\square$

9. PARTIAL SUM OPERATORS

A key application of the Littlewood-Paley theory are dyadic decompositions. Those in turn use partial sum operators, which we describe in this section.

Definition 9.1. If $\rho \subset \mathbb{R}^n$ is a rectangle whose sides are parallel to the axes, we define the operator $S_\rho : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ implicitly by

$$\widehat{S_\rho f} = \mathbf{1}_\rho \widehat{f}.$$

This operator is certainly bounded on L^2 by Plancherel's theorem, but in fact we can extend this to the L^p setting.

Theorem 9.2. *If $f \in L^2 \cap L^p$, $1 < p < \infty$, and $\rho \subset \mathbb{R}^n$ is a rectangle as above, then*

$$\|S_\rho f\|_{L^p} \leq A_p \|f\|_{L^p}$$

with A_p independent of ρ .

We will actually prove a stronger theorem:

Theorem 9.3. *Let $\mathcal{H}$ be the space of ℓ^2 summable complex sequences, and let $\mathcal{R} = \{\rho_j\}_{j=1}^\infty$ be an arbitrary sequence of rectangles in $\mathbb{R}^n$ whose sides are parallel to the axes. For $f \in L^2(\mathbb{R}^n, \mathcal{H})$ define*

$$S_{\mathcal{R}}(f) = (S_{\rho_1}(f_1), S_{\rho_2}(f_2), \dots).$$

If $1 < p < \infty$ and $f \in L^2 \cap L^p(\mathbb{R}^n, \mathcal{H})$, then

$$\|S_{\mathcal{R}}(f)\|_{L^p} \leq A_p \|f\|_{L^p}.$$

Sketch of proof. There are four main steps in the proof.

Step 1. We prove the theorem under the assumption $n = 1$ and that each ρ_j is the same interval $(-\infty, 0)$. Recall the Hilbert transform

$$Hf = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} dy,$$

whose multiplier is $m_H(x) = i \operatorname{sign} x$. Then $\frac{\operatorname{Id} + iH}{2}$ has multiplier $\mathbf{1}_{(-\infty, 0)}$, so $S_{(-\infty, 0)} = \frac{\operatorname{Id} + iH}{2}$. The claim will follow from the following claim:

Claim. *Let $f = (f_1, f_2, \dots) \in L^2 \cap L^p(\mathbb{R}^n, \mathcal{H})$, and set $\tilde{H}f(x) = (Hf_1(x), Hf_2(x), \dots)$. Then $\|\tilde{H}f\|_{L^p} \leq A_p \|f\|_{L^p}$.*

Proof of claim. If $K(x) = \operatorname{Id}_{\ell^2} \frac{1}{\pi x}$, then $\tilde{H}f = K * f$. On the other hand $K(x)$ satisfies the L^p theory of singular integrals that we have already discussed. $\square$

Step 2. We now relax the assumption that all the rectangles be $(-\infty, 0)$, and instead require that they be $\rho_i = (-\infty, a_i)$ for $i = 1, 2, \dots$. Then the theorem easily follows from step 1, provided we know how to handle shifts from $(-\infty, a)$ to $(-\infty, 0)$. Indeed,

$$f(x) \widehat{e^{-2\pi x a}}(\xi) = \widehat{f}(\xi - a).$$

Step 3. For arbitrary dimensions $n \geq 1$ we begin by assuming that the ρ_i are halfspaces, $\rho_i = \{x \in \mathbb{R}^n : x_1 < a_i\}$. The thing to note is that the indicator function really acts on one coordinate so we can effectively try to reduce to the one-dimensional case.

Let $S_{(-\infty, a_i)}^{(1)}$ denote the operator on $L^2(\mathbb{R}^n)$ acting only on the x_1 variable, i.e.

$$S_{(-\infty, a_i)}^{(1)} f(x_1, \dots, x_n) = \int_{\mathbb{R}} \widehat{f}^{\xi_1}(\xi_1, x_2, \dots, x_n) \mathbf{1}_{(-\infty, a_i)}(\xi_1) e^{-2\pi i x_1 \xi_1} d\xi_1.$$

We claim that $S_{(-\infty, a_i)}^{(1)} = S_{\rho_i}$ for all i , $f \in L^2$. This is obvious when f decomposes as $f(x_1, x_2, \dots, x_n) = g(x_1)h(x_2, \dots, x_n)$, or when it is a linear combination of such things. On the other hand those functions span a dense subset of L^2 , so by continuity and step 2 we can conclude $\|S_{\mathcal{R}} f\|_{L^p} \leq A_p \|f\|_{L^p}$.

Step 4. The general case, with arbitrary n and arbitrary rectangles. This actually follows immediately from step 3, since every rectangle in $\mathbb{R}^n$ that we consider is the intersection of $2n$ halfspaces. Apply step 3 a total of $2n$ times. $\square$

Remark 9.4. One may naturally wonder whether the same result holds true when rectangles are replaced by other simple objects in $\mathbb{R}^n$. Charles Fefferman notoriously constructed a counterexample to the assertion when the cutoff function is a disk instead of a rectangle [?]. More precisely,

$$\widehat{T}f = \mathbf{1}_B \widehat{f}$$

does *not* define a bounded operator $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, unless $p = 2$ or $n = 1$.

10. DYADIC DECOMPOSITION

Definition 10.1. We define the dyadic decomposition of $\mathbb{R}$ to be

$$\mathbb{R} = \{0\} \cup \left(\bigcup_{k \in \mathbb{Z}} [2^k, 2^{k+1}] \right) \cup \left(\bigcup_{k \in \mathbb{Z}} [-2^{k+1}, -2^k] \right).$$

The collection of intervals $[2^k, 2^{k+1}]$, $[-2^{k+1}, -2^k]$ above is called Δ (or Δ^1 when highlighting the dimension).

Correspondingly we can decompose $\mathbb{R}^n$ in a similar way by first decomposing our axes as described, and then considering the rectangles that form. The corresponding collection Δ (or Δ^n) then consists of elements $\rho = I_{m_1} \times \dots \times I_{m_n}$ where $I_{m_i} = [2^{m_i}, 2^{m_i+1}]$ or $[-2^{m_i+1}, -2^{m_i}]$.

The first observation one makes is that, for all n ,

$$\text{Id} = \sum_{\rho \in \Delta} S_{\rho} \text{ in } L^2(\mathbb{R}^n)$$

and since $\mathbf{1}_{\rho_1} \mathbf{1}_{\rho_2} = 0$ when $\rho_1 \neq \rho_2$,

$$\|f\|_{L^2}^2 = \sum_{\rho \in \Delta} \|S_{\rho} f\|_{L^2}^2.$$

When $p \neq 2$ the situation is more complicated.

Theorem 10.2. *Let $f \in L^2 \cap L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then*

$$B_p \|f\|_{L^p} \leq \|(\sum_{\rho \in \Delta} |S_\rho f|^2)^{1/2}\|_{L^p} \leq A_p \|f\|_{L^p}.$$

Remark 10.3. By the standard duality argument it suffices to show either one of the two inequalities above. For example, if we manage to show the rightmost inequality

$$(10.1) \quad \|(\sum_{\rho \in \Delta} |S_\rho f|^2)^{1/2}\|_{L^p} \leq A_p \|f\|_{L^p},$$

then we can obtain the inequality on the left by observing that:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f \bar{g} \right| &= \left| \int_{\mathbb{R}^n} \sum_{\rho} S_\rho f \overline{S_\rho g} \right| \leq \|(\sum_{\rho \in \Delta} |S_\rho f|^2)^{1/2}\|_{L^p} \|(\sum_{\rho \in \Delta} |S_\rho g|^2)^{1/2}\|_{L^q} \\ &\leq A_p \|(\sum_{\rho \in \Delta} |S_\rho f|^2)^{1/2}\|_{L^p} \|g\|_{L^q}, \end{aligned}$$

then taking the supremum over all $g \in L^2 \cap L^q$, and then using the $(L^q)^* \cong L^p$ duality.

The proof of (10.1) is going to make extensive use of the Rademacher function

$$r_0(t) = \begin{cases} 1 & \text{on } [0, 1/2], \\ 0 & \text{on } (1/2, 1) \end{cases}$$

extended periodically on $\mathbb{R}$ and $r_m(t) = r_0(2^m t)$. Note that the r_m are pairwise orthogonal in $L^2[0, 1]$. If $\{a_m\}_m \in \ell^2$ then we can define

$$F(t) = \sum_{m=0}^{\infty} a_m r_m(t), \quad t \in [0, 1].$$

By orthogonality it's simple to see that $\|F\|_{L^2[0,1]}^2 = \sum_m |a_m|^2$. We're going to employ the following useful fact:

$$(10.2) \quad c_p \|F\|_{L^p} \leq \|F\|_{L^p} \leq \tilde{c}_p \|F\|.$$

The n dimensional analog of the Rademacher function uses the family of functions $r_m : \mathbb{R}^n \rightarrow \mathbb{R}$, $m \in \mathbb{N}^n$, defined by $r_m(t) = r_{m_1}(t_1) \cdots r_{m_n}(t_n)$. If $\{a_m\}_{m \in \mathbb{N}^n}$ is ℓ^2 summable, then we may still define

$$F(t) = \sum_{m \in \mathbb{N}^n} a_m r_m(t), \quad t \in Q = [0, 1]^n,$$

which satisfies the corresponding estimate

$$\|F\|_{L^p} \approx \|F\|_{L^2} = \left(\sum_m |a_m|^2 \right)^{1/2}.$$

Proof of Theorem 10.2, $n = 1$. Let Δ^1 be the family of dyadic intervals in $\mathbb{R}$. Instead of looking at S_ρ we look at the mollified operators $\widetilde{S}_\rho$ defined implicitly by

$$\widehat{\widetilde{S}_\rho f} = \varphi_\rho \widehat{f}$$

where φ is a smooth function, $\varphi \equiv 1$ on $[1, 2]$, $\varphi \equiv 0$ outside $[0, 4]$, and $0 \leq \varphi \leq 1$ elsewhere, and $\varphi_\rho(x) = \varphi(2^{-k}x)$ when $\rho = [2^k, 2^{k+1}]$. Note that $S_\rho = S_\rho \widetilde{S}_\rho$.

For $t \in [0, 1]$ define

$$\widetilde{T}_t = \sum_m r_m(t) \widetilde{S}_{I_m}$$

whose multiplier is

$$\widetilde{m}_t(x) = \sum_m r_m(t) \varphi_{I_m}(x).$$

Fixing x , there exist at most three nonzero terms in the sum above and in fact we can bound $|\widetilde{m}_t(x)| + |x| |\partial_x \widetilde{m}_t(x)| \leq B$. By applying 8.12 we know that $\widetilde{T}_t : L^p \rightarrow L^p$, $1 < p < \infty$, is bounded independently of t . Therefore

$$\begin{aligned} A_p^p \|f\|_{L^p}^p &\geq \int_0^1 \|\widetilde{T}_t f\|_{L^p}^p dt = \int_0^1 \int_{\mathbb{R}^n} \left| \sum_m r_m(t) \widetilde{S}_{I_m} f(x) \right|^p dx dt \\ &\geq c_p \int_{\mathbb{R}^n} \left(\sum_m |\widetilde{S}_{I_m} f(x)|^2 \right)^{p/2} dx. \end{aligned}$$

The last step used estimate (10.2). By taking p -th roots,

$$\left\| \left(\sum_m |\widetilde{S}_{I_m} f|^2 \right)^{1/2} \right\|_{L^p} \leq A'_p \|f\|_{L^p}.$$

Notice that we are almost done, except we have shown the result for $\widetilde{S}f$ instead of Sf . Note that $S_{I_m} \widetilde{S}_{I_m} = S_{I_m}$, so if we apply theorem 9.3 to the map

$$F : x \mapsto \{\widetilde{S}_{I_m} f(x)\}_m \in \ell^2$$

and the collection of rectangles $\mathcal{R} = \{I_m\}_m$, then

$$\left\| \left(\sum_{\rho \in \Delta} |S_\rho f|^2 \right)^{1/2} \right\|_{L^p} = \|S_{\mathcal{R}} F\|_{L^p} \leq \left\| \left(\sum_m |\widetilde{S}_{I_m} f|^2 \right)^{1/2} \right\|_{L^p} \leq A'_p \|f\|_{L^p},$$

which completes the proof when $n = 1$. $\square$

The difficulty in the case $n = 1$ was the fact that we had to go through mollified partial sum operators in order to prove (10.1). Now we can just treat that as a black box and avoid the mollified operators altogether. In accordance with this, let's set

$$T_t f(x) = \sum_{m \in \mathbb{N}^n} r_m(t) S_{I_m} f(x)$$

as I_m runs through the dyadic rectangles, and r_m runs through the n -dimensional Rademacher functions. By applying the left hand side inequality of Theorem 10.2 for $n = 1$ to $T_t f$

$$\|T_t f\|_{L^p} \leq c_p \left\| \left(\sum_{\rho \in \Delta^1} |S_\rho T_t f|^2 \right)^{1/2} \right\|_{L^p}$$

then by the orthogonality of the Rademacher functions

$$\leq c_p \left\| \left(\sum_{\rho \in \Delta^1} |S_\rho f|^2 \right)^{1/2} \right\|_{L^p}$$

and finally by the right hand side inequality of Theorem (10.2) for $n = 1$ to f

$$\leq c_p \|f\|_{L^p},$$

or in other words, $\|T_t f\|_{L^p} \leq c_p \|f\|_{L^p}$ for $n = 1$.

Proof of 10.2, arbitrary n . First we extend this last inequality to arbitrary n . Let $T_{t_1}^{(1)}$ be the operator above acting on the first coordinate, i.e.

$$T_{t_1}^{(1)} f(x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} r_{m_1}(t_1) S_{I_{m_1}}^{(x_1)} f(x_1, \dots, x_n).$$

By the inequality above,

$$\int_{\mathbb{R}} |T_{t_1}^{(1)} f(x_1, \dots, x_n)|^p dx_1 \leq c_p \int_{\mathbb{R}} |f(x_1, \dots, x_n)|^p dx_1.$$

Integrating over $x_2, \dots, x_n$ too gives $\|T_{t_1}^{(1)} f\|_{L^p} \leq c_p \|f\|_{L^p}$. Follow the same steps for each of the remaining coordinates sequentially on the functions

$$T_{t_1}^{(1)} f(x_1, \dots, x_n) \rightsquigarrow T_{t_2}^{(2)} T_{t_1}^{(1)} f(x_1, \dots, x_n) \rightsquigarrow \dots \rightsquigarrow T_{t_n}^{(n)} \dots T_{t_1}^{(1)} f(x_1, \dots, x_n) = T_t f,$$

the last equality holding true by definition of the Rademacher functions in higher dimensions. Then by iterating the single-variable estimate,

$$\|T_t f\|_{L^p} = \|T_{t_n}^{(n)} \dots T_{t_1}^{(1)} f\|_{L^p} \leq \dots \leq c_p \|T_{t_1}^{(1)} f\|_{L^p} \leq c_p \|f\|_{L^p}.$$

Integrating over all $t \in Q = [0, 1]^n$,

$$\begin{aligned} c_p \|f\|_{L^p}^p &\geq \int_Q \int_{\mathbb{R}^n} |T_t f(x)|^p dx dt = \int_{\mathbb{R}^n} \int_Q \left| \sum_m r_m(t) S_{I_m} f(x) \right|^p dt dx \\ &= \int_{\mathbb{R}^n} \|F^{(x)}(t)\|_{L^p(Q)}^p dx. \end{aligned}$$

By the special properties of this F -function described prior to the proof of the $n = 1$ case, the L^p and L^2 norms are comparable. As such,

$$c_p \|f\|_{L^p}^p \geq \int_{\mathbb{R}^n} \|F^{(x)}(t)\|_{L^2(Q)}^p dx = \int_{\mathbb{R}^n} \left(\sum_{\rho} |S_\rho f(x)|^2 \right)^{p/2} dx,$$

which is the required result upon taking p -th roots. $\square$

This essentially concludes our discussion of dyadic decompositions. The takeaway point is that when we want to use sharp cutoff functions we need to work harder in proving our theorems by mollifying first. In fact, not all sharp cutoff functions are bound to work.

In practice it might be a better idea to use smooth cutoff functions, because we can apply our previous singular integral multiplier theory on them. Consider a smooth function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\phi \geq 0, \quad \phi \equiv \text{const on } B_1, \quad \phi \equiv 0 \text{ outside } B_2,$$

and for which $\psi(x) = \phi(x) - \phi(2x)$ satisfies

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) \equiv 1, \quad \xi \neq 0.$$

Then this smooth cutoff function provides a dyadic, radial, annular decomposition of the form

$$\text{Id} = \sum_{j=-\infty}^{\infty} \Delta_j f, \quad \text{where } \widehat{\Delta_j f}(\xi) = \psi(2^{-j}\xi) \widehat{f}(\xi).$$

This decomposition does satisfy the equivalent of Theorem 10.2:

$$\|(\sum_j |\Delta_j f|^2)^{1/2}\|_{L^p} \approx \|f\|_{L^p}.$$

11. BOURGAN-BRESIZ INEQUALITY

We start with a threefold motivational excursion.

Part I. Suppose you're looking to solve

$$\text{div } \mathbf{Y} = f$$

for a given $f \in L^n(\mathbb{R}^n)$. Clearly you solve this in the class $\mathbf{Y} \in \dot{W}^{1,n}(\mathbb{R}^n) = \{|\nabla Y_i| \in L^n\}$ because you can first solve $\Delta u = f$, get $f \in L^n(\mathbb{R}^n) \Rightarrow \nabla u \in \dot{W}^{1,n}(\mathbb{R}^n)$ by elliptic regularity, and set $\mathbf{Y} = \nabla u$.

Question: can you solve this in the class $\mathbf{Y} \in L^\infty(\mathbb{R}^n)$? Remember, Sobolev embedding fails in the critical case $p = n$ so that's of no assistance here.

Part II. The counterpart of Sobolev embedding in the global space $\mathbb{R}^n$ that replaces $W^{1,n}$ with $\dot{W}^{1,n}$ is:

Proposition 11.1 (Gagliardo-Nirenberg). *For any $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\|u\|_{L^{n/(n-1)}} \leq c_n \|Du\|_{L^1}.$$

Part III. As remarked, $W^{1,n}(\mathbb{R}^n) \not\hookrightarrow L^\infty(\mathbb{R}^n)$, but the following theorem provides a remedy for the situation:

Theorem 11.2 (van Schaftingen). *If $\mathbf{f}, \mathbf{g}$ are compactly supported vector fields on $\mathbb{R}^n$ and $\text{div } \mathbf{f} = 0$, then*

$$\left| \int_{\mathbb{R}^n} \mathbf{f} \cdot \mathbf{g} \right| \leq c_n \|\mathbf{f}\|_{L^1} \|\nabla \mathbf{g}\|_{L^n}.$$

Of course this would be trivial if there were an embedding $\dot{W}^{1,n} \hookrightarrow L^\infty$, but there isn't one.

Now let's return to Part I. The question asked is answered in the affirmative by the following:

Proposition 11.3 (Bourgain-Brezis). *There exists $c_n > 0$ such that for any $f \in L^n(\mathbb{R}^n)$ we can find $\mathbf{Y} \in L^\infty \cap \dot{W}^{1,n}(\mathbb{R}^n)$ with $\operatorname{div} \mathbf{Y} = f$, $\|\mathbf{Y}\|_{L^\infty} \leq c_n \|f\|_{L^n}$.*

It turns out that Propositions 11.3 and 11.1 are equivalent.

Proof of Proposition 11.3 $\Rightarrow$ Proposition 11.1. We estimate the $L^{n/n-1}$ norm of u by duality. For $f \in L^n(\mathbb{R}^n)$ arbitrary, by 11.3 there exists $\mathbf{Y} \in L^\infty \cap \dot{W}^{1,n}(\mathbb{R}^n)$ with $\operatorname{div} \mathbf{Y} = f$. Use of the weak definition of $\operatorname{div} \mathbf{Y} = f$ yields

$$\left| \int u f \right| = \left| \int u \operatorname{div} \mathbf{Y} \right| \leq \|\nabla u\|_{L^1} \|\mathbf{Y}\|_{L^\infty} \leq c_n \|\nabla u\|_{L^1} \|f\|_{L^n}.$$

Since $f \in L^n(\mathbb{R}^n)$ were arbitrary, $\|u\|_{L^{n/n-1}} \leq c_n \|\nabla u\|_{L^1}$. $\square$

Proof of Proposition 11.1 $\Rightarrow$ Proposition 11.3. Let $(L^1)^n$ denote the space of vector fields on $\mathbb{R}^n$ with L^1 components, and consider its subspace $E = \{\nabla u \in (L^1)^n : u \in C_c^\infty(\mathbb{R}^n)\}$. Given $f \in L^n(\mathbb{R}^n)$ we may define the linear operator $T : E \rightarrow \mathbb{R}$

$$T(\nabla u) \triangleq - \int u f.$$

By Hölder's inequality and then by Proposition 11.1,

$$|T(\nabla u)| \leq \|u\|_{L^{n/n-1}} \|f\|_{L^n} \leq c_n \|\nabla u\|_{L^1} \|f\|_{L^n},$$

so $T : E \rightarrow \mathbb{R}$ is a bounded linear operator with $\|T\| \leq c_n \|f\|_{L^n}$. By Hahn-Banach we can extend this to a bounded linear operator $T : (L^1)^n \rightarrow \mathbb{R}$ with $\|T\| \leq c_n \|f\|_{L^n}$. By $(L^1)^* = L^\infty$ duality there exists $\mathbf{Y} \in (L^\infty)^n$ such that

$$T(\mathbf{v}) = \int \mathbf{Y} \cdot \mathbf{v} \quad \forall \mathbf{v} \in (L^1)^n.$$

By construction, $\|\mathbf{Y}\|_{L^\infty} = \|T\| \leq c_n \|f\|_{L^n}$, and

$$- \int u f = T(\nabla u) = \int \mathbf{Y} \cdot \nabla u \quad \forall u \in C_c^\infty(\mathbb{R}^n)$$

so $\operatorname{div} \mathbf{Y} = u$ in the required weak sense. $\square$

In fact, in the above proposition we may view the vector field $\mathbf{Y}$ as a 1-form and $\operatorname{div} \mathbf{Y}$ as d^*Y . Then 11.3 admits the following generalization:

Theorem 11.4. *Suppose $l \neq n - 1$. Then for any $(l + 1)$ -form X on $\mathbb{R}^n$ with coefficients in $\dot{W}^{1,n}$, there exists an $(l + 1)$ -form Y with coefficients in L^∞ such that*

$$d^*Y = d^*X$$

Recall that 11.3 is equivalent to 11.1. By the same token, 11.4 is equivalent to the following generalization of 11.1 due to L. Lanzani and E. Stein:

Theorem 11.5. *Suppose u is a smooth l -form with compact support in $\mathbb{R}^n$.*

(a) *If $l \neq 1$ or $n - 1$, then*

$$(11.1) \quad \|u\|_{L^{n/n-1}} \leq C(\|du\|_{L^1} + \|d^*u\|_{L^1})$$

(b) If $l = 1$ or $n - 1$, then

$$(11.2) \quad \|u\|_{L^{n/n-1}} \leq C(\|du\|_{\mathcal{H}^1} + \|d^*u\|_{\mathcal{H}^1})$$

where $\mathcal{H}^1$ denotes the Hardy space.

We will next show that 11.5 (and thus 11.4) is equivalent to 11.2. First we state an equivalent form of 11.2:

Theorem 11.6. *Let $\mathbf{f} = (f_1, \dots, f_n)$ be a compactly supported vector field on $\mathbb{R}^n$ with $\operatorname{div} \mathbf{f} = 0$, and let g be a compactly supported scalar function. Then*

$$(11.3) \quad \int_{\mathbb{R}^n} f_1 g \leq C \|f_1\|_{L^1} \|\nabla g\|_{L^n}$$

Proof of 11.6 (or 11.2) $\Rightarrow$ 11.5. We'll only prove part (a). Note that when $l = 0$ or n , (11.1) reduces to the usual Gagliardo-Nirenberg inequality.

Next suppose $1 < l < n - 1$. Let u, φ be smooth l -forms with compact support on $\mathbb{R}^n$. We write

$$(11.4) \quad \begin{aligned} (u, \varphi) &= (u, (dd^* + d^*d)\Delta^{-1}\varphi) \\ &= (du, d\Delta^{-1}\varphi) + (d^*u, d^*\Delta^{-1}\varphi) \\ &= \int_{\mathbb{R}^n} \sum_I (du)_I (d\Delta^{-1}\varphi)_I dx + \int_{\mathbb{R}^n} \sum_J (d^*u)_J (d^*\Delta^{-1}\varphi)_J dx \end{aligned}$$

where I and J run through the set of non-decreasing $(l+1)$ - and $(l-1)$ -tuples, respectively.

For each $(l+1)$ -tuple $I = (i_1, \dots, i_{l+1})$ in the first summation, since $l+1 < n$, there exists $i \in \{1, \dots, n\}$ such that i is not in I . Next recall that

$$0 = (d^2u)_{i, i_1, \dots, i_{l+1}} = \partial_i (du)_I - \sum_{\nu=1}^{l+1} \partial_{i_\nu} (du)_{I_\nu}$$

where I_ν is obtained by replacing i_ν with i . Therefore for each I we can find a divergence-free vector field of which $(du)_I$ is one of the components. This allows us to apply 11.6 and obtain

$$(11.5) \quad |(du, d\Delta^{-1}\varphi)| \leq C \|du\|_{L^1} \|\nabla d\Delta^{-1}\varphi\|_{L^n} \leq C \|du\|_{L^1} \|\varphi\|_{L^n}$$

where the last inequality follows from the estimates on singular integral operators.

Noting that $l > 1$ and $(d^*)^2 = 0$, we can estimate the second summation in the last line of (11.4) using the same idea and get

$$(11.6) \quad |(d^*u, d^*\Delta^{-1}\varphi)| \leq C \|d^*u\|_{L^1} \|\varphi\|_{L^n}.$$

Combining (11.4), (11.5) and (11.6), we get

$$|(u, \varphi)| \leq C(\|du\|_{L^1} + \|d^*u\|_{L^1}) \|\varphi\|_{L^n}$$

Since φ is arbitrary, we conclude that

$$\|u\|_{L^{n/n-1}} \leq C(\|du\|_{L^1} + \|d^*u\|_{L^1})$$

□

Before showing the reverse implication, we give a proof of 11.6. We'll need the following lemma.

Lemma 11.7. *Let Φ be a smooth compactly supported function on $\mathbb{R}^N$. Given $p > N$ and $\delta > 0$, there exists a decomposition of Φ , $\Phi = \Phi_1 + \Phi_2$, such that*

$$(11.7) \quad \begin{aligned} \|\Phi_1\|_{L^\infty} &\leq C\delta^{1-N/p}\|\nabla\Phi\|_{L^p} \\ \|\nabla\Phi_2\|_{L^\infty} &\leq C\delta^{-N/p}\|\nabla\Phi\|_{L^p} \end{aligned}$$

Proof. Consider the annular Littlewood-Paley decomposition $\Phi = \sum_j \Delta_j \Phi$, as described in the concluding remarks of the previous section. Next we fix $M > 0$ such that $2^M \approx \delta^{-1}$ and write

$$\begin{aligned} \Phi_1 &= \sum_{j>M} \Delta_j \Phi \text{ (high frequency)} \\ \Phi_2 &= \sum_{j\leq M} \Delta_j \Phi \text{ (low frequency)} \end{aligned}$$

To proceed we'll make use of the following Bernstein inequality:

Claim 11.8. *For $1 \leq p \leq q \leq \infty$, we have*

$$(11.8) \quad \|\Delta_j f\|_{L^q} \leq C2^{jN(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}$$

Proof of Claim. Recall that $\Delta_j f$ is defined by

$$\widehat{\Delta_j f}(\xi) = \psi\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)$$

Inverting the Fourier transform and letting $\psi = \hat{K}$, we get

$$(11.9) \quad \Delta_j f = 2^{jN} K(2^j \cdot) * f$$

(11.8) now follows from an application of the Young's inequality. $\square$

We continue with the proof of 11.7. Recall that ψ is smooth and supported in $B_2 - B_{1/2}$, so the following function is smooth

$$\varphi(\xi) = \frac{\psi(\xi)}{2\pi i \xi}$$

Let G be defined by $\hat{G} = \varphi$. Then

$$\hat{K}(\xi) = \psi(\xi) = 2\pi i \xi \varphi(\xi) = 2\pi i \xi \hat{G}(\xi) = \widehat{\nabla G}(\xi)$$

Hence $K = \nabla G$. Using (11.9) and integrating by parts, we have

$$\begin{aligned} \Delta_j \Phi(x) &= \int_{\mathbb{R}^N} 2^{jN} K(2^j y) f(x-y) dy \\ &= 2^{-j} \int_{\mathbb{R}^N} 2^{jN} G(2^j y) \nabla f(x-y) dy \end{aligned}$$

An application of Young's inequality as in the proof of (11.8) now gives

$$\|\Delta_j \Phi\|_{L^\infty} \leq C 2^{-j(1-N/p)} \|\nabla \Phi\|_{L^p}$$

Summing over $j > M$, we get

$$\|\Phi_1\|_{L^\infty} \leq C 2^{-M(1-N/p)} \|\nabla \Phi\|_{L^p} \leq C \delta^{1-N/p} \|\nabla \Phi\|_{L^p}$$

This proves the first inequality in (11.7).

Next for each $j \leq M$ we apply (11.8) to $\nabla \Phi$, getting

$$\|\Delta_j \nabla \Phi\|_{L^\infty} \leq C 2^{jN/p} \|\nabla \Phi\|_{L^p}$$

Summing over $j \leq M$, we get

$$\|\nabla \Phi_2\|_\infty \leq C 2^{MN/p} \|\nabla \Phi\|_{L^p} \leq C \delta^{-N/p} \|\nabla \Phi\|_{L^p}$$

□

Proof of 11.6. Write $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. We introduce the following notation:

$$\Phi^{x_1}(x') = \Phi(x_1, x')$$

Fixing $x_1 \in \mathbb{R}$, apply 11.7 with $N = n - 1$ and $p = n$. Then we get $\Phi^{x_1} = \Phi_1^{x_1} + \Phi_2^{x_1}$ with

$$(11.10) \quad \begin{aligned} \|\Phi_1^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C \delta^{1/n} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})} \\ \|\nabla' \Phi_2^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C \delta^{-\frac{n-1}{n}} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})} \end{aligned}$$

where ∇' denotes gradient in the x' -variable. Next let $\mathbf{f} = (f_1, \dots, f_n)$ and Φ be as given in the statement of 11.6 and consider

$$\int_{\mathbb{R}^{n-1}} f_1^{x_1} \Phi^{x_1} dx' = \int_{\mathbb{R}^{n-1}} f_1^{x_1} \Phi_1^{x_1} dx' + \int_{\mathbb{R}^{n-1}} f_1^{x_1} \Phi_2^{x_1} dx' \equiv I + II$$

By Hölder and (11.10),

$$|I| \leq C \delta^{1/n} \|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}$$

As for II , we have

$$\begin{aligned} |II| &= \left| \int_{\mathbb{R}^{n-1}} \left[\int_{-\infty}^{x_1} \partial_t f_1^t(x') dt \right] \Phi_2^{x_1} dx' \right| \quad (\text{fundamental theorem of Calculus}) \\ &= \left| \int_{\mathbb{R}^{n-1}} \left[\int_{-\infty}^{x_1} - \sum_{i=2}^n \partial_i f_i^t(x') dt \right] \Phi_2^{x_1} dx' \right| \quad (\text{div } \mathbf{f} = 0) \end{aligned}$$

Integrating by parts, applying Hölder and using (11.10), we get

$$|II| \leq \|f\|_{L^1(\mathbb{R}^n)} \|\nabla' \Phi_2^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \delta^{-\frac{n-1}{n}} \|f\|_{L^1(\mathbb{R}^n)} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}$$

Now we choose δ such that

$$\|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})} \delta^{1/n} = \|f\|_{L^1(\mathbb{R}^n)} \delta^{\frac{n-1}{n}}$$

Then we conclude that

$$\left| \int_{\mathbb{R}^{n-1}} f_1^{x_1} \Phi^{x_1} dx' \right| \leq |I| + |II|$$

$$\leq C \|f\|_{L^1(\mathbb{R}^n)}^{1/n} \|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}^{\frac{n-1}{n}} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}$$

Finally, we integrate the above inequality in x_1

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_1 \Phi dx \right| &= \left| \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} f_x^{x_1} \Phi^{x_1} dx' \right] dx_1 \right| \\ &\leq C \|f\|_{L^1(\mathbb{R}^n)}^{1/n} \int_{\mathbb{R}} \|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}^{\frac{n-1}{n}} \|\nabla' \Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})} dx_1 \\ &\leq C \|f\|_{L^1(\mathbb{R}^n)} \|\nabla \Phi\|_{L^n(\mathbb{R}^n)} \quad (\text{H\"older}) \end{aligned}$$

and we're done. $\square$

Proof of 11.4 $\implies$ 11.2. Let $\mathbf{f}, \mathbf{g}$ be vector fields on $\mathbb{R}^n$, such that $\operatorname{div} \mathbf{f} = 0$. Interpret $\mathbf{f}$ and $\mathbf{g}$ as $(n-1)$ -forms f, g with $df = 0$. Write $g = d\alpha + d^*\beta$, and we have that

$$(f, g) = (f, d\alpha) + (f, d^*\beta) = (f, d\alpha)$$

By 11.4 we can find an $(n-1)$ -form $\Psi \in L^\infty$ such that

$$\begin{cases} d^*\Psi = d^*(d\alpha) \\ \|\Psi\|_{L^\infty} \leq C \|d^*(d\alpha)\|_{L^n} = C \|d^*g\|_{L^n} \end{cases}$$

and hence $d\alpha = \Psi + d^*\gamma$ for some γ . We therefore have

$$|(f, g)| = |(f, d\alpha)| = |(f, \Psi)| \leq \|f\|_{L^1} \|\Psi\|_{L^\infty} \leq C \|f\|_{L^1} \|d^*g\|_{L^n}$$

which completes the proof. $\square$

In fact one can prove more delicate versions of 11.4, 11.5 and 11.2

Theorem 11.9. (B-B) Given any $(l+1)$ -form X with coefficients in $\dot{W}^{1,n}\mathbb{R}^n$ ($l \neq n-1$), we can find an $(l+1)$ -form Y , with coefficients in $\dot{W}^{1,n} \cap L^\infty$ such that

$$\begin{cases} d^*X = d^*Y \\ \|Y\|_{L^\infty} + \|\nabla Y\|_{L^n} \leq C \|\nabla X\|_{L^n} \end{cases}$$

Theorem 11.10. (B-B) Given any l -form u ($l \neq 1, n-1$), with coefficients in $C_c^\infty(\mathbb{R}^n)$, then

$$\|u\|_{L^{\frac{n}{n-1}}} \leq C (\|d^*u\|_{L^1 + \dot{W}^{-1, \frac{n}{n-1}}} + \|du\|_{L^1 + \dot{W}^{-1, \frac{n}{n-1}}})$$

Theorem 11.11. (B-B) If $\mathbf{f}, \mathbf{g}$ are vector fields on $\mathbb{R}^n$, with coefficients in $C_c^\infty(\mathbb{R}^n)$ and $\operatorname{div} \mathbf{f} = 0$, then

$$\left| \int \mathbf{f} \cdot \mathbf{g} \right| \leq \|\mathbf{f}\|_{L^1 + \dot{W}^{-1, \frac{n}{n-1}}} + \|\nabla \mathbf{g}\|_{L^n}$$

Note 11.12. Recall that if B_1, B_2 are Banach spaces, then $B_1 \cap B_2$ and $B_1 + B_2$ are Banach spaces, with norms

$$\begin{aligned} \|b\|_{B_1 \cap B_2} &= \|b\|_{B_1} + \|b\|_{B_2} \\ \|b\|_{B_1 + B_2} &= \inf \{ \|f\|_{B_1} + \|g\|_{B_2} : b = f + g \text{ for } f \in B_1, g \in B_2 \} \end{aligned}$$

And further, $(B_1 \cap B_2)^* = B_1^* + B_2^*$. We make the definition

$$\dot{W}^{-1, \frac{n}{n-1}} := (\dot{W}^{1, n})^*$$

Proving the equivalence of these three is relatively straightforward, following the same ideas as before. The main lemma towards proving 11.9 is the following.

Lemma 11.13 (Main Lemma). *Given $\delta > 0$, we can find an $A_\delta > 0$ such that for every $f \in \dot{W}^{1, n}$, there exists an $F \in \dot{W}^{1, n} \cap L^\infty$ approximating f all but one direction, in the sense that*

$$\begin{cases} \sum_{i=2}^n \|\partial_i(f - F)\|_{L^n} \leq \delta \|\nabla f\|_{L^n} \\ \|F\|_{L^\infty} + \|\nabla F\|_{L^n} \leq A_\delta \|\nabla f\|_{L^n} \end{cases}$$

(note the sum misses the first index)

Proof of 11.13 $\implies$ 11.9. Here is the key idea of the proof: when computing d^* of a $(q+1)$ -form (provided $q < n-1$), in each component some index remains uninvolved.

Take a $(q+1)$ -form X in $\dot{W}^{1, n}(\wedge^{q+1}, \mathbb{R}^n)$. We can find an $\alpha^{(0)} \in \dot{W}^{1, n}(\wedge^{q+1}, \mathbb{R}^n)$ such that

$$d^* \alpha^{(0)} = d^* X \quad \text{and} \quad \|\nabla \alpha^{(0)}\|_{L^n} \leq C \|d^* X\|_{L^n}$$

For given multi-index I of length $q+1 < n$, there is an $i \notin I$ and $\beta_I^{(0)}$ approximating $\alpha_I^{(0)}$ in all but the i^{th} direction:

$$\begin{cases} \|\sum_{j \neq i} \partial_j(\beta_I^{(0)} - \alpha_I^{(0)})\|_{L^n} \leq \delta \|\nabla \alpha_I^{(0)}\|_{L^n} \\ \|\beta_I^{(0)}\|_{L^\infty} + \|\nabla \beta_I^{(0)}\|_{L^n} \leq A_\delta \|\nabla \alpha_I^{(0)}\|_{L^n} \end{cases}$$

which implies that for an appropriate choice of δ we have

$$\begin{cases} \|d^*(X - \beta^{(0)})\|_{L^n} \leq C\delta \|d^* X\|_{L^n} = 1/2 \|d^* X\|_{L^n} \\ \|\beta^{(0)}\|_{L^\infty} + \|\nabla \beta^{(0)}\|_{L^n} \leq A \|d^* X\|_{L^n} \end{cases}$$

Repeat with $X - \beta^{(0)}$ in place of X , and deduce the existence of a sequence $\beta^{(k)}$ in $\dot{W}^{1, n} \cap L^\infty$ such that

$$\|\beta^{(k)}\|_{L^\infty} + \|\nabla \beta^{(k)}\|_{L^n} \leq 2^{-k} A \|d^* X\|_{L^n}$$

for every k , and

$$\|d^*(X - \sum_{k=1}^K \beta^{(k)})\|_{L^n} \leq 2^{-K-1} \|d^* X\|_{L^n}.$$

Let $Y = \sum \beta^{(k)}$, and the theorem follows. $\square$

We work towards proving 11.13. We make use of the algebraic relation

$$1 = \sum a_j \prod_{1 \leq j' < j} (1 - a_{j'}) + \prod (1 - a_j)$$

which holds for infinitely sums provided each $a_j \in [0, 1]$. A probabilistic interpretation of the above can be found in flipping coins of probabilities a_j : each summand in the RHS is the probability of getting a "heads" after precisely j flips.

Recall the Littlewood-Paley decomposition: $f = \sum \Delta_j f =: \sum f_j$. If we can find functions G_j such that $|f_j| \leq G_j \leq 1$, then the sum

$$\sum f_j \prod (1 - G_j) \in L^\infty$$

one could then try setting F (as in the lemma) to be this sum. We find

$$\begin{aligned} f - F &= \sum_j f_j (1 - \prod_{j' > j} (1 - G_{j'})) \\ &= \sum_j f_j \sum_{j' > j} G_{j'} \prod_{1 < j'' < j'} (1 - G_{j''}) \\ &= \sum_{j'} G_{j'} \sum_j f_j \prod_{1 < j'' < j} (1 - G_{j''}) \\ &= \sum_j G_j H_j \end{aligned}$$

defining $H_j = \sum_j f_j \prod_{1 < j'' < j} (1 - G_{j''}) \leq 1$. We then estimate

$$\begin{aligned} |\partial_i(f - F)| &\leq \sum_j |\partial_i G_j| + |\partial_i H_j| \\ &\leq \sum_j \left(|\partial_i G_j| + \sum_{j' < j} |\partial_i f_{j'}| + |\partial_i G_{j'}| \right) \end{aligned}$$

Ultimately we need to bound $\|\sum_j \partial_i G_j\|_n$ in terms of $\|\nabla f\|_n$. This won't hold with the naive choice $G_j = |\Delta_j f|$.

Here are two possible strategies:

(1) Control the low frequencies by the high frequencies. Replace $2^j |\Delta_j f|$ by

$$2^j |\Delta_j f| \chi_{\{2^j |\Delta_j f| > \sum_{k < j} 2^k |\Delta_k f|\}}$$

in which case

$$\begin{aligned} \|\sum_j (2^j |\Delta_j f| \chi_{\{\dots\}})\|_{L^n} &\leq \|\sup_j 2^j |\Delta_j f|\|_{L^n} \\ &\leq \|(\sum_j (2^j |\Delta_j f|)^2)^{1/2}\|_{L^n} \\ &\leq \|\nabla f\|_{L^n} \end{aligned}$$

(2) Control the spatial directions. Choose functions ω_j with

$$|\Delta_j f| \leq \omega_j \leq \|\Delta_j f\|_{L^\infty}$$

such that

$$\begin{aligned} |\partial_i \omega_j| &\leq 2^{j-\sigma} \omega_j \text{ for } i = 2, \dots, n \\ |\partial_1 \omega_j| &\leq 2^j \omega_j \end{aligned}$$

Taking $G_j = \omega_j$, we just need to control

$$\|\sup 2^j \omega_j\|_{L^n} \leq 2^{\sigma \frac{n-1}{n}} \|\nabla f\|_{L^n}$$

12. ISOPERIMETRIC INEQUALITIES IN NONPOSITIVE CURVATURE

Let M^n be a simply connected manifold with non-positive sectional curvature.

Conjecture 12.1. The isoperimetric inequality is valid: For all bounded domains $\Omega \subseteq M$, we have

$$\text{vol}(\Omega) \leq C(n) \text{area}(\partial\Omega)^{n/(n-1)},$$

where $C(n)$ is the constant for round balls in $\mathbb{R}^n$, i.e. $C(n) = \text{vol}(B) / \text{area}(\partial B)^{n/(n-1)}$.

In dimension $n = 2$, this was proved by Weil; in dimension $n = 3$, there are various approaches by Kleiner [?], Ritoré [?] and Schulze [?] (the latter uses a flow by a power of the mean curvature scalar). We will present Kleiner's approach later in this section. First, we will discuss Croke's proof which works in all dimensions but is only sharp for $n = 4$. Let us also mention that in a similar direction, Hoffman and Spruch proved a Michael-Simon inequality for non-positive sectional curvature, namely if $\Sigma^n \subset M^{n+1}$, then

$$\left(\int_{\Sigma} |u|^{n/(n-1)} d\sigma \right)^{(n-1)/n} \leq C \left(\int_{\Sigma} |u| \cdot |\vec{H}| + |\nabla u| d\sigma \right).$$

The idea of Croke's proof in $n = 4$ dimensions is to represent the volume and area of $\Omega \subset M$ by the integral over the unit tangent bundle of $\partial\Omega$. We change our notation to match the one of Croke:

- $(M^n, \partial M^n)$ is a compact set in a manifold with non-positive sectional curvature,
- UM denotes the unit tangent bundle of M ,
- $U\partial M$ denotes the unit tangent bundle of M over ∂M , i.e. $U\partial M = U_{\partial M}M$,
- $U^+\partial M$ denotes the unit upper hemisphere in $U\partial M$. The measure on $U^+\partial M$ is the local product measure du where the measure of the fiber is that of the unit upper hemisphere;
- for $v \in UM$, γ_v is the geodesic with $\gamma'_v(0) = v$, and we put

$$\ell(v) := \max\{t \mid \gamma_v(t) \in M\},$$

so that $\gamma_v(\ell(v)) \in \partial M$;

- for $p \in \partial M$ and $u \in U_p^+\partial M$, let $\cos(u)$ denote the cosine of the angle between u and n_p , the inward unit normal to ∂M at p , i.e. $\cos(u) = \langle u, n_p \rangle$.

Lemma 12.2. (Santalo) For any integrable function f on UM , we have

$$(12.1) \quad \int_{UM} f(v) dv = \int_{U^+\partial M} \int_0^{\ell(u)} f(\gamma_u(t)) \cos(u) dt du.$$

Lemma 12.3. (1) $\text{vol}(M) = \frac{1}{\omega_{n-1}} \int_{U^+\partial M} \ell(u) \cos(u) du.$

- (2) If $\text{ant}(u)$ denotes the opposite vector at the end of the geodesic starting at $u \in U^+ \partial M$, i.e. $\text{ant}(u) = -\gamma'_u(\ell(u))$, then

$$(12.2) \quad \int_{U^+ \partial M} g(u) \cos(u) du = \int_{U^+ \partial M} g(\text{ant}(u)) \cos(u) du.$$

Proof. For the first part, simply use $f \equiv 1$ in Lemma 12.2. For the second part, note that (12.1) means that the geodesic flow ξ is a measure-preserving map from $Q = \{(u, t) \mid u \in U^+ \partial M, t \in [0, \ell(u)]\}$ to UM , where Q is given the measure $\cos(u) dt du$, and ξ has an inverse (smooth almost everywhere), also measure-preserving, which is evidently given by $\xi^{-1}(v) = (-\gamma'_{-v}(\ell(-v)), \ell(-v))$ for $v \in UM$. Since the antipodal map $-1: UM \rightarrow UM$ is also measure-preserving, we see that

$$\xi^{-1} \circ (-1) \circ \xi: (u, t) \mapsto (\text{ant}(u), \ell(u) - t)$$

is also measure-preserving. But this means that for every integrable $G: Q \rightarrow \mathbb{R}$, we have

$$\int_{U^+ \partial M} \int_0^{\ell(u)} G(u, t) \cos(u) dt du = \int_{U^+ \partial M} \int_0^{\ell(u)} G(\text{ant}(u), \ell(u) - t) \cos(u) dt du,$$

so plugging in $G(u, t) = g(u)/\ell(u)$ and integrating out t (note that $\ell(u) = \ell(\text{ant}(u))$), we obtain (12.2). $\square$

Lemma 12.4. (1) *We have*

$$\int_{U^+ \partial M} \frac{\ell(u)^{n-1}}{\cos(\text{ant } u)} du \leq \text{area}(\partial M)^2,$$

and equality holds iff M is flat and convex.

- (2) *We have*

$$\int_{U^+ \partial M} \cos(\text{ant } u)^{1/(n-2)} \cos(u)^{(n-1)/(n-2)} du \leq \text{vol}(\partial M) c_2(n),$$

for some universal constant $c_2(n)$, with equality iff $\cos u = \cos(\text{ant } u)$ for all $u \in U^+ \partial M$.

Proof. For the first inequality, let dx denote the volume form of M and dp the volume form of ∂M . Use polar normal coordinates (u, r) at $q \in \partial M$ and consider $\exp\{tu \mid u \in U_q^+ \partial M, t \in [0, \ell(u)]\}$, then $dx = F(u, r) du dr$, where F is the Jacobian; now since M is non-positively curved, comparison tells us that $F(u, r) \geq r^{n-1}$. But clearly

$$\int_{U_q^+ \partial M} \frac{F(u, \ell(u))}{\cos(\text{ant } u)} du_q = \text{area}(A_q),$$

where $A_q \subseteq \partial M$ (with equality holding for all q iff ∂M is convex), and integrating this over ∂M yields

$$\int_{U^+ \partial M} \frac{F(u, \ell(u))}{\cos(\text{ant } u)} du \leq \text{area}(\partial M)^2.$$

Using $F(u, \ell(u)) \geq \ell(u)^{n-1}$ finishes the proof of the first part.

We next prove the second inequality: Using Hölder's inequality and Lemma 12.3, we estimate

$$\begin{aligned}
& \int_{U+\partial M} \cos(\text{ant } u)^{1/(n-2)} \cos(u)^{(n-1)/(n-2)} du \\
&= \int_{U+\partial M} \cos(\text{ant } u)^{1/(n-2)} \cos(u)^{1/(n-2)} \cos(u) du \\
&\leq \left(\int_{U+\partial M} \cos(\text{ant } u)^{2/(n-2)} \cos(u) du \right)^{1/2} \left(\int_{U+\partial M} \cos(u)^{2/(n-2)} \cos(u) du \right)^{1/2} \\
&= \int_{U+\partial M} \cos(u)^{n/(n-2)} \cos(u) du \\
&= \int_{\partial M} \left(\int_{U_q^+ \partial M} \cos(u)^{n/(n-2)} du_q \right) dq \\
&= \text{vol}(\partial M) c_2(n),
\end{aligned}$$

where

$$c_2(n) = \int_{U_q^+ \partial M} \cos(u)^{n/(n-2)} du$$

(which is of course independent of q). Equality holds iff $\cos(\text{ant } u) = k \cos(u)$. Since $\cos(u), \cos(\text{ant } u) \in [0, 1]$, and both attain the value 1 at some point, we must have $k = 1$. $\square$

We are now ready to prove the isoperimetric inequality:

Proof of the isoperimetric inequality, sharp for $n = 4$. Using Hölder and Lemma 12.4, we estimate

$$\begin{aligned}
\text{vol}(M) &= \frac{1}{\omega_{n-1}} \int_{U+\partial M} \ell(u) \cos(u) du \\
&= \frac{1}{\omega_{n-1}} \int \frac{\ell(u)}{\cos(\text{ant } u)^{1/(n-1)}} \cos(\text{ant } u)^{1/(n-1)} \cos(u) du \\
&\leq \frac{1}{\omega_{n-1}} \left(\int \frac{\ell(u)^{n-1}}{\cos(\text{ant } u)} du \right)^{1/(n-1)} \left(\int \cos(\text{ant } u)^{\frac{n-1}{n-2} \cdot \frac{1}{n-1}} \cos(u)^{\frac{n-1}{n-2}} du \right)^{\frac{n-2}{n-1}} \\
&\leq \frac{1}{\omega_{n-1}} \text{area}(\partial M)^{2/(n-1)} (\text{area}(\partial M) c_2(n))^{(n-2)/(n-1)} \\
&= \frac{c_2(n)^{(n-2)/(n-1)}}{\omega_{n-1}} \text{area}(\partial M)^{n/(n-1)}.
\end{aligned}$$

In order to have equality, we get

$$\frac{\ell(u)^{n-1}}{\cos(\text{ant } u)} = \lambda \cos(\text{ant } u)^{1/(n-2)} \cos(u)$$

which for $n = 4$ means $\ell(u)^3 = \lambda \cos(\text{ant } u) \cos(\text{ant } u)^{1/2} \cos(u)^{3/2}$, and from the case of equality in Lemma 12.4, we also know $\cos(\text{ant } u) = \cos(u)$; hence $\ell(u) =$

$\lambda^{1/3} \cos(u)$ when $n = 4$, which is exactly the equation of a ball with diameter $\lambda^{1/3}$ in Euclidean space. (Note that flatness follows from the first part of Lemma 12.4.)

Let us remark that in the above proof, if $n \neq 4$, we would get that M is flat and $\cos(u) = \cos(\text{ant } u)$ in the case of equality still, but this would give $\ell(u) = \tilde{\lambda} \cos(u)^{2/(n-2)}$; such M however does not exist. $\square$

Next, we present Kleiner's approach to the isoperimetric inequality in $n = 3$ dimensions. Recall that we have the isoperimetric profile function

$$I_M(V) = \inf\{\text{area}(\partial\Omega) : \Omega \subset M \text{ compact, } C^\infty, \text{ vol}(\Omega) = V\}.$$

If Ω minimizes $I_M(V)$ for $V = \text{vol}(\Omega)$ we call Ω an isoperimetric domain. We're going to need the following result on the existence and regularity of isoperimetric domains in bounded sets, which we will treat as a black box.

Theorem 12.5. *Let B^n be a compact manifold with smooth boundary ∂B^n , and let $V \in (0, \text{vol}(B^n))$. Then there exists a domain $\Omega \subset B^n$ with boundary $\Sigma = \partial\Omega$ such that:*

- (1) $\text{vol}(\Omega) = V$, $\text{area}(\Sigma) = I_B(V)$,
- (2) (when $n = 3$,) Σ is $C^{1,1}$ in a neighborhood of ∂B^n ,
- (3) there exists a singular set $\Sigma_{\text{sing}} \subset \Sigma \cap \text{int } B$ of Hausdorff dimension $\leq n - 8$ such that $(\Sigma \cap \text{int } B) \setminus \Sigma_{\text{sing}}$ is a C^∞ hypersurface with constant mean curvature H ,
- (4) the mean curvature h of Σ is defined a.e. and $h \leq H$.

Remark 12.6. The first and third claims are standard in geometric measure theory. The second claim pertaining to boundary regularity is due to [?]. The last claim follows by a variational argument. Suppose u is Σ -deformation supported in the interior of B , and v is a Σ -deformation supported on ∂B , and such that $\int_\Sigma v = \int_\Sigma u$ to preserve volume. By minimality,

$$\int_\Sigma nHu \geq \int_\Sigma nhv.$$

Since H is constant where u is supported,

$$nH \geq \frac{\int_\Sigma nhv}{\int_\Sigma u} = \frac{\int_\Sigma nhv}{\int_\Sigma v}$$

for all v . In view of v being arbitrary and the differentiation theorem, $H \geq h$.

Remark 12.7. $I_B(\cdot)$ is continuous. In other words, if Ω_i is a sequence of isoperimetric domains with $\text{vol}(\Omega_i) \rightarrow V$, then $\text{area}(\partial\Omega_i) \rightarrow I_B(V)$.

Theorem 12.8. *Let M^3 be simply connected, complete, noncompact, with $\text{sec} \leq C < 0$. Then its isoperimetric profile function satisfies $I_{M^3} \geq I_{\mathbb{H}_C^3}$.*

The theorem is going to follow from the following proposition. In fact the proposition implies the theorem in all dimensions, but we can only prove the proposition when $n = 3$.

Proposition 12.9. *Let M^3 be simply connected, complete, noncompact, with $\sec \leq C < 0$. Let Ω be compact, $C^{1,1}$, $\Sigma = \partial\Omega$. Then*

$$\max_{\Sigma} H \geq H_{\mathbb{H}_C^3}(\text{area}(\Sigma)),$$

where $H_{\mathbb{H}_C^3}(\text{area}(\Sigma))$ is the mean curvature of the geodesic ball of area equal to $\text{area}(\Sigma)$ in the model hyperbolic three-space $\mathbb{H}_C^3$ with curvature C .

Proposition $\Rightarrow$ Theorem, arbitrary n . Fix $\Omega \subset M^3$. Choose a geodesic ball B large enough that $\Omega \subset B \subset M^n$. Let Ω_V be an isoperimetric domain in B with $\text{vol}(\Omega_V) = \text{vol}(\Omega) \triangleq V$. Let H_V be the constant mean curvature of Ω_V on its interior regular part. Then by the proposition, $H_V \geq H_{\mathbb{H}_C^n}(\text{area}(\partial\Omega_V))$.

Choose a volume-decreasing deformation of Ω_V supported on the interior regular part. Call the deformed surfaces $\Omega_{V+\Delta V}$, for $\Delta V \leq 0$, and $\text{vol}(\Omega_{V+\Delta V}) = V + \Delta V$. By minimization, $I_B(V + \Delta V) \leq \text{area}(\partial\Omega_{V+\Delta V})$, so for $\Delta V < 0$:

$$\frac{I_B(V + \Delta V) - I_B(V)}{\Delta V} \geq \frac{\text{area}(\partial\Omega_{V+\Delta V}) - \text{area}(\partial\Omega_V)}{\Delta V} \rightarrow nH_V \geq nH_{\mathbb{H}_C^n}(I_B(V)),$$

i.e. the left derivative of the isoperimetric profile satisfies $D_- I_B(V) \geq nH_{\mathbb{H}_C^n}(I_B(V))$. On the other hand in the model case we have $I'_{\mathbb{H}_C^n}(V) = nH_{\mathbb{H}_C^n}(I_{\mathbb{H}_C^n}(V))$. The two isoperimetric profiles agree when $V = 0$ so by integrating the differentials,

$$\text{area}(\partial\Omega) \geq \text{area}(\partial\Omega_V) = I_B(V) \geq I_{\mathbb{H}_C^n}(V).$$

□

Proof of proposition, $n = 3$. First we do the case where $\Sigma = \partial\Omega$ is a topological $\mathbb{S}^2$. By Gauss-Bonnet and the Gauss equation,

$$4\pi = \int_{\Sigma} K = \int_{\Sigma} \sec_{T\Sigma} + k_1 k_2 \leq \int_{\Sigma} C + H^2 \leq (C + \max_{\Sigma} H^2) \text{area}(\Sigma).$$

We would get exact equality in hyperbolic space, so $\max_{\Sigma} H \geq H_{\mathbb{H}_C^3}(\text{area}(\Sigma))$.

Let's check the rigidity claim when $\Sigma = \partial\Omega$. If equality holds above then $\sec_{T\Sigma} \equiv C$, which is also the upper bound for sectional curvatures on M^3 , and $\Sigma = \partial\Omega$ is umbillic, whose mean curvature matches the mean curvature of the appropriate geodesic ball in hyperbolic space. These conditions force Ω to be a geodesic ball in $\mathbb{H}_C^3$.

Now let's look at general domains Ω . Let D_0 be the convex hull of Ω , and let $D_s = \{x \in M : \text{dist}(x, D_0) < s\}$ be the s -fattening of D_0 . Then at least for $s > 0$ small enough, D_s is also convex, $C_s = \partial D_s$ is a topological $\mathbb{S}^2$, and there exists a sufficiently smooth closest point projection $r : M^3 \setminus D_0 \rightarrow \Sigma$. By Gauss-Bonnet,

$$\begin{aligned} 4\pi &= \int_{C_s} K = \int_{C_s} \sec_{T\Sigma} + k_1 k_2(C_s) \\ &\leq \int_{C_s} C + k_1 k_2(C_s) \\ &= \int_{r_s^{-1}(\Sigma \cap \partial D_0)} C + k_1 k_2(C_s) + \int_{C_s \setminus r_s^{-1}(\Sigma \cap \partial D_0)} C + k_1 k_2(C_s) \end{aligned}$$

$$\begin{aligned} &\leq \int_{r_s^{-1}(\Sigma \cap \partial D_0)} C + H_{C_s}^2 + C \operatorname{area}(C_s \setminus r_s^{-1}(\Sigma \cap \partial D_0)) \\ &\quad + C \int_{C_s \setminus r_s^{-1}(\Sigma \cap \partial D_0)} k_1 k_2(C_s). \end{aligned}$$

The middle term is ≤ 0 since $C < 0$. We let $s \downarrow 0$. In doing that, the first integral can be estimated from above by $(C + H_{C_0}^2) \operatorname{area}(\Sigma \cap \partial D_0)$ because the Riccati equation gives controlled growth rate bounds on H . The last integral converges to zero, since on the regular points of $\partial D_0 \setminus \Sigma$ we have $k_1 k_2 = 0$ because we're on the part of the convex hull of Σ that lies away from Σ . Putting it all together,

$$4\pi \leq (C + H_0^2) \operatorname{area}(\Sigma \cap \partial D_0) \leq (C + H_0^2) \operatorname{area}(\Sigma).$$

Like we argued before, if we were on hyperbolic space we would have gotten equality all along. Therefore, $H_0 \geq H_{\mathbb{H}_C^3}(\operatorname{area}(\Sigma))$. $\square$

Now we give the proof of the 3-dimensional isoperimetric inequality due to Ritoré. Let M^3 be a complete, simply-connected 3-manifold with non-positive sectional curvature and let Σ be an embedded compact surface.

Pick a point $p \in \Sigma$ and let $d_p(\cdot)$ be the distance to p . Consider the following conformal change of metric

$$g_\varepsilon = \rho_\varepsilon^2 g = e^{2u_\varepsilon} g, \text{ where } u_\varepsilon = \log \frac{2\varepsilon}{1 + \varepsilon^2 d^2}$$

We will prove the following inequality:

$$(12.3) \quad \int_{\Sigma} H^2 d\sigma \geq 4\pi$$

with equality iff Ω is flat. Note that (12.3) implies 12.8, for if (12.3) holds, then we have

$$\operatorname{area}(\Sigma)(\max_{\Sigma} H)^2 \geq \int_{\Sigma} H^2 d\sigma \leq 4\pi = \operatorname{area}(\Sigma)(H_{\mathbb{H}_0^3}(\operatorname{area}(\Sigma)))^2$$

and thus 12.9 holds. As shown previously, this implies 12.8.

Proof of (12.3). Recalling that the sectional curvature of a 2-plane σ in TM before and after the conformal change of metric are related by

$$e^{2u_\varepsilon} K_\varepsilon(\sigma) = K(\sigma) - \nabla^2 u_\varepsilon(e_i, e_i) - \nabla^2 u_\varepsilon(e_j, e_j) + (\nabla_{e_i} u_\varepsilon)^2 + (\nabla_{e_j} u_\varepsilon)^2 - |\nabla u_\varepsilon|^2$$

where $\{e_i, e_j\}$ is an orthonormal basis for σ with respect to g . Plugging in the expression for u_ε and applying the Hessian comparison theorem (M has non-positive sectional curvature), we arrive at

$$K_\varepsilon e^{2u_\varepsilon} \geq K + e^{2u_\varepsilon}$$

To continue, we first write

$$\int_{\Sigma} H^2 d\sigma = \int_{\Sigma} (H^2 + K) d\sigma - \int_{\Sigma} K d\sigma.$$

Note that K denote the sectional curvature of $T\Sigma$ with respect to the ambient metric g and NOT the Gauss curvature of Σ .

Next we write

$$\begin{aligned} H^2 + K &= H^2 - \kappa_1 \kappa_2 + \kappa_1 \kappa_2 + K \\ &= |\overset{\circ}{A}|^2 + (\kappa_1 \kappa_2 + K) \end{aligned}$$

then we see that

$$\int_{\Sigma} (H^2 + K) d\sigma$$

is conformally invariant. Hence

$$\begin{aligned} \int_{\Sigma} H^2 d\sigma &= \int_{\Sigma} (H^2 + K) d\sigma - \int_{\Sigma} K d\sigma \\ &= \int_{\Sigma} (H_{\varepsilon}^2 + K_{\varepsilon}) d\sigma_{\varepsilon} - \int_{\Sigma} K d\sigma \\ &= \int_{\Sigma} H_{\varepsilon}^2 d\sigma_{\varepsilon} + \int_{\Sigma} (e^{2u_{\varepsilon}} K_{\varepsilon} - K) d\sigma \\ &\geq \int_{\Sigma} H_{\varepsilon}^2 d\sigma_{\varepsilon} + \int_{\Sigma} d\sigma_{\varepsilon} \\ &\geq \int_{\Sigma} d\sigma_{\varepsilon} \end{aligned}$$

Using the fact that letting $\varepsilon \rightarrow 0$ corresponds to blow-up at p with a spherical metric, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma} d\sigma_{\varepsilon} = 4\pi$$

Hence (12.3), and thus 12.8, is proved $\square$

Next let's turn our attention to compact manifolds and discuss Levy and Gromov's isoperimetric inequality. For a smooth domain Ω in the standard sphere S^n , define the following function

$$I_{S^n}(\alpha) = \frac{\text{vol}(B_{\alpha})}{\text{vol}(S^n)},$$

where B_{α} is a geodesic ball with $\text{vol}(B_{\alpha}) = \alpha \text{vol}(S^n)$. Then the classical isoperimetric inequality says that

$$\text{area}(\partial\Omega) \geq \text{vol}(S^n) I_{S^n}(\alpha), \text{ where } \alpha = \frac{\text{vol}(\Omega)}{\text{vol}(S^n)}$$

In 1919, P. Levy generalized this result to convex hypersurfaces in $\mathbb{R}^{n+1}$. Later, Gromov extended Levy's method to all Riemannian manifolds with a lower bound on the Ricci curvature [?].

Theorem 12.10. *Let M^n be a closed manifold with $\text{Ric} \geq (n-1)g$. Then for any smooth domain Ω in M , we have*

$$\text{area}(\partial\Omega) \geq \text{vol}(M^n) I_{S^n}(\alpha), \quad \alpha = \frac{\text{vol}(\Omega)}{\text{vol}(M^n)}$$

The proof relies on a comparison theorem due to Levy, Heintz and Karcher, which we state below. Let H^{n-1} be a smooth hypersurface of M and define the following map:

$$\begin{aligned} \exp : H \times \mathbb{R}_+ &\rightarrow M \\ (h, t) &\mapsto \exp_h(t\nu) \end{aligned}$$

where ν is a unit normal vector field. Let $J(h, t)$ denote the Jacobian of this map. Next consider a model pair $(\overline{M}, \overline{H})$ where $\overline{M}$ has constant sectional curvature k and $\overline{H}$ has constant mean curvature η . We denote the corresponding Jacobian by $\overline{J}$.

There's an explicit formula for $\overline{J} = \overline{J}_{\eta, k}$, namely

$$\overline{J}_{\eta, k}(h, t) = \left(\frac{dS_k(t)}{dt} - \eta S_k(t) \right)^{n-1}, \quad S_k(t) = \frac{1}{\sqrt{k}} \sin \sqrt{k}t$$

With these notations, the Levy-Heintz-Karcher comparison theorem says the following.

Theorem 12.11. *With the above notations, supposing in addition that $\text{Ric}_M \geq (n-1)kg$ and that the mean curvature of H at h with respect to ν is bounded below by η , we have*

$$|J(h, t)| \leq \overline{J}_{\eta, k}(t)$$

Proof of 12.10. Fix $\alpha \in (0, 1)$, and consider all the hypersurfaces in M that divides it into two parts, Ω and M/Ω , with

$$\text{vol}(\Omega) = \alpha \text{vol}(M)$$

A theorem of Almgren guarantees the existence of a hypersurface $\partial\Omega$ in this class having constant mean curvature, which we denote by η . (Strictly speaking, the $\partial\Omega$ produced by Almgren's theorem may have a singular set, but it does not affect our arguments.)

For $h \in \partial\Omega$, let $c(h)$ be the distance to the first focal point along the normal geodesic $\exp_h(t\nu)$ and assume that ν points into Ω , then

$$(12.4) \quad \text{vol}(\Omega) = \int_{\partial\Omega} \int_0^{c(h)} J(h, t) dt d\sigma$$

Combining 12.11 and (12.4), we get

$$(12.5) \quad \text{vol}(\Omega) \leq \text{area}(\partial\Omega) \int_0^{\beta_{\eta, 1}^+} \overline{J}_{\eta, 1} dt$$

where $\beta_{\eta, 1}^+$ is the first positive time when $\overline{J}_{\eta, 1} = 0$. Note that if we take Ω to be a geodesic ball in S^n of radius $\beta_{\eta, 1}^+$ then we would get equality in (12.5). Therefore

$$\text{area}(\partial\Omega) \geq \text{vol}(\partial\Omega) \frac{A_1(\beta_{\eta, 1}^+)}{V_1(\beta_{\eta, 1}^+)}$$

where $A_1(r)(V_1(r))$ denote the area(volume) of $\partial B_r(B_r)$ in S^n . Note that A_1/V_1 is decreasing in r .

Now if the radius of B_α , $r(B_\alpha)$, is greater than or equal to $\beta_{\eta,1}^+$, then

$$\begin{aligned} \text{area}(\partial\Omega) &\geq \text{vol}(\Omega) \frac{A_1(r(B_\alpha))}{V_1(r(B_\alpha))} \\ &= \frac{\text{vol}(\Omega)}{\text{vol}(B_\alpha)} \text{area}(\partial B_\alpha) \\ &= \frac{\text{vol}(M)}{\text{vol}(S^n)} \text{area}(\partial B_\alpha) \\ &= \text{vol}(M^n) I_{S^n}(\alpha) \end{aligned}$$

and we are done. On the other hand, if $r(B_\alpha) < \beta_{\eta,1}^+$, then we reverse the normal vector field so that it points into M/Ω and redefine J and $\bar{J}_{\eta,1}$ accordingly. Letting $\beta_{\eta,1}^-$ be the first time when $\bar{J}_{\eta,1}(t) = 0$, we must have $r(B_\alpha) \geq \beta_{\eta,1}^-$ by assumption. Now the argument from the previous case applies. $\square$