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Minimal surfaces and the Allen–Cahn equation on 3-manifolds

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Abstract

Following our lecture at the 2024 International Congress of Basic Science, we discuss the context and main results of our joint work with O. Chodosh, "Minimal surfaces and the Allen-Cahn equation on 3-manifolds." We also discuss recent progress, open questions, and future directions.

Contents

1 Introduction

Our main object of study are **minimal submanifolds** of Riemannian manifolds, which are variational generalizations of the notion of a geodesic.

Definition 1.1 *A closed immersed submanifold* $\Sigma^k \subset (M^n, g)$ *is called* ***minimal*** *if it is a critical point of the area functional, i.e.,*

$$\left[\tfrac{d}{dt}\operatorname{Area}_g(\operatorname{graph}_\Sigma(tX))\right]_{t=0} = 0.$$

for all $X \in \Gamma(N\Sigma)$ *(normal vector fields to* Σ*).*

We further define the (Morse) ***index*** *and* ***nullity*** *of* Σ *as:*

$$\begin{aligned}\operatorname{ind}[\Sigma] = \max\{\dim V : V \subset \Gamma(N\Sigma) \textit{ subspace with} \\ \left[\tfrac{d^2}{dt^2}\operatorname{Area}_g(\operatorname{graph}_\Sigma(tX))\right]_{t=0} < 0 \textit{ for all } X \in V \setminus \{0\}\},\\ \operatorname{nul}[\Sigma] = \max\{\dim V : V \subset \Gamma(N\Sigma) \textit{ subspace with} \\ \left[\tfrac{d^2}{dt^2}\operatorname{Area}_g(\operatorname{graph}_\Sigma(tX))\right]_{t=0} = 0 \textit{ for all } X \in V\}.\end{aligned}$$

Minimal submanifolds have had spectacular applications in Riemannian geometry, helping resolve open questions that, a priori, do not involve them in their formulation. We list a small number of important examples from classical differential geometry.

Conjecture 1.2 (Frankel's conjecture) *Closed Kähler manifolds with positive bisectional curvature are biholomorphic to complex projective spaces.*

Siu–Yau proved Conjecture 1.2 in [1] relying crucially on constructing suitable homotopically area-minimizing (branched) two-spheres. There is a separate, algebraic proof of the conjecture by Mori ([2]).

Conjecture 1.3 (Geroch's conjecture) *There is no metric with positive scalar curvature on a 3-torus.*

Schoen–Yau proved Conjecture 1.3 in [3] relying crucially on constructing suitable area-minimizing incompressible surfaces. There is another proof by Gromov–Lawson using spinors ([4]), and a recent one by Stern using harmonic maps ([5]).

Conjecture 1.4 *Ricci flow with surgery terminates in finite time on closed orientable 3-manifolds with only on-aspherical factors in their prime decomposition.*

After Perelman's initial proof of Conjecture 1.4 in [6], Colding–Minicozzi gave a new elegant proof in [7] by keeping track of the area of certain minimal two-spheres obtained by mountain pass methods and showing it must become zero in finite time.

In each example above, it was crucial to have a sufficiently powerful existence theory for minimal submanifolds. We refer the reader to [8] for a comprehensive list of references and history.

2 Allen–Cahn solutions

Our aim is to discuss a recent way to obtain minimal hypersurfaces: as limits, in the Hausdorff sense, of nodal sets of solutions to certain semilinear elliptic equations.

Definition 2.1 *Let* $W : \mathbf{R} \to \mathbf{R}$ *be a smooth double-well potential, i.e., with:*

- $W > 0$ *on* $\mathbf{R} \setminus \{\pm 1\}$, $W(\pm 1) = 0$,

- *$W''(\pm 1) > 0 > W''(0)$,*
- *$W' \neq 0$ on $\mathbf{R} \setminus \{\pm 1, 0\}$, $W'(\pm 1) = W'(0) = 0$,*
- *W even.*

Fix a closed (M^n, g). For $\varepsilon > 0$, consider $E_\varepsilon : C^\infty(M) \to \mathbf{R}$ given by

$$E_\varepsilon[u] := \int_M \tfrac{\varepsilon}{2} |\nabla_g u|^2 + \tfrac{1}{\varepsilon} W(u) \, d\mu_g.$$

*We say that $u_\varepsilon \in C^\infty(M)$ is an **Allen–Cahn ε-solution** if it is a critical point of E_ε, i.e.,*

$$\left[\tfrac{d}{dt} E_\varepsilon[u_\varepsilon + t\zeta]\right]_{t=0} = 0$$

for all $\zeta \in C^\infty(M)$; equivalently, u_ε satisfies the Allen–Cahn equation

$$\varepsilon^2 \Delta_g u_\varepsilon = W'(u_\varepsilon) \text{ on } M.$$

*We define the (Morse) **index** and **nullity** as before:*

$$\begin{aligned}
\operatorname{ind}_\varepsilon[u_\varepsilon] = \max\{&\dim V : V \subset C^\infty(M) \text{ subspace with} \\
&\left[\tfrac{d^2}{dt^2} E_\varepsilon[u_\varepsilon + t\zeta]\right]_{t=0} < 0 \text{ for all } \zeta \in V \setminus \{0\}\}, \\
\operatorname{nul}_\varepsilon[u_\varepsilon] = \max\{&\dim V : V \subset C^\infty(M) \text{ subspace with} \\
&\left[\tfrac{d^2}{dt^2} E_\varepsilon[u_\varepsilon + t\zeta]\right]_{t=0} = 0 \text{ for all } \zeta \in V\}.
\end{aligned}$$

For much of the rest of the write-up, we fix a particular choice of W, such as the canonical choice

$$W(t) = \tfrac{1}{4}(1 - t^2)^2.$$

We will seek to study how nodal sets of Allen–Cahn ε-solutions converge to minimal hypersurfaces as $\varepsilon \to 0$, also keeping track of multiplicity. To that end, the following definition seems convenient:

Definition 2.2 *Fix a closed (M^n, g) and a closed immersed hypersurface $\Sigma \subset M$. We say that a sequence of Allen–Cahn ε_i-solutions u_{ε_i} with $\varepsilon_i \to 0$ satisfies*

$$u_{\varepsilon_i} \rightsquigarrow \Sigma \text{ as } \varepsilon_i \to 0$$

if $\{u_{\varepsilon_i} = 0\} \to \Sigma$ in the Hausdorff sense and $E_{\varepsilon_i}[u_{\varepsilon_i}] \to \operatorname{Area}_g(\Sigma)$ as $i \to \infty$.[1]

This convergence of Allen–Cahn ε_i-solutions to minimal hypersurfaces was first studied by Modica, Mortola, and Sternberg ([9, 10, 11]) in the setting of minimizers and Γ-convergence. It has seen explosive growth in the last twenty years, and we give a brief outline of results that contextualize and lead the way to ours.

Pacard–Ritoré showed in [12] that generic minimal hypersurfaces must occur as Allen–Cahn ε-solution limits; see also the works of De Philippis–Pigati ([13]) and del Pino–Kowalczyk–Wei ([14]).

[1] We are suppressing a W-dependent multiplicative constant in front of $\operatorname{Area}_g(\Sigma)$.

Theorem 2.3 ([12]) *Fix a closed (M^n, g) and a closed embedded hypersurface $\Sigma \subset M$ which is:*

- *nondegenerate (i.e.,* $\operatorname{nul}[\Sigma] = 0$*), and*
- *separating (i.e., it bounds a domain in M).*

Then, for all small $\varepsilon > 0$, there exist Allen–Cahn ε-solutions u_ε such that $u_\varepsilon \rightsquigarrow \Sigma$ as $\varepsilon \to 0$.

It was recently observed by the author in [15] that the assumption of nondegeneracy cannot be removed in general; see also related work of Caju–Gaspar ([16]) and Chen–Gaspar ([17]). This can be viewed as an example of extra rigidity on the part of the Allen–Cahn equation.

Naturally, if we are looking to the Allen–Cahn equation to provide us with a means to construct *new* minimal hypersurfaces, then Theorem 2.3 goes in the wrong direction: it *assumes* the existence of a minimal hypersurface and produces Allen–Cahn ε-solutions converging to it. The Gaspar–Guaraco $\mathbf{Z}_2$-cohomological min-max construction theorem below from [18] offers Allen–Cahn ε-solutions without starting from a background Σ; see also the previous work of Guaraco ([19]):

Theorem 2.4 ([18]) *Fix a closed (M^n, g). For $\varepsilon > 0$, $p \in \mathbf{N}$, denote by $c_\varepsilon(p)$ the p-parameter $\mathbf{Z}_2$-cohomological min-max energy of E_ε on (M, g). Then:*

(a) $0 = E_\varepsilon(\pm 1) = c_\varepsilon(0) < \ldots \le c_\varepsilon(p) \le \ldots \le E_\varepsilon(0) = \frac{1}{\varepsilon} \operatorname{Vol}_g(M) W(0)$,

(b) if $c_\varepsilon(p) < E_\varepsilon(0)$, then there exists an Allen–Cahn ε-solution $u_{\varepsilon,p}$ with
$$E_\varepsilon[u_{\varepsilon,p}] = c_\varepsilon(p), \ \operatorname{ind}_\varepsilon[u_{\varepsilon,p}] \le p \le \operatorname{ind}_\varepsilon[u_{\varepsilon,p}] + \operatorname{nul}_\varepsilon[u_{\varepsilon,p}],$$

(c) $\lim_{\varepsilon \to 0} c_\varepsilon(p) = A_{M,g}(p) \in (0, \infty)$*, and*

(d) $\alpha'_{M,g} \le A_{M,g}(p) p^{-\frac{1}{n}} \le \alpha''_{M,g}$*, where* $\alpha'_{M,g}, \alpha''_{M,g} \in (0, \infty)$.

This theorem was strongly influenced by the Almgren–Pitts min-max theory ([20]) and its developments by Marques–Neves ([21]). Subsequently, the authors obtained in [22] a refinement of (d) mirroring the analogous Almgren–Pitts Weyl law of Liokumovich–Marques–Neves ([23]):

Theorem 2.5 ([22]) *Assume the setting of Theorem 2.4. Then:*

(d') ("Allen–Cahn Weyl Law") $\lim_{p \to \infty} A_{M,g}(p) p^{-\frac{1}{n}} = \alpha(n) \operatorname{Vol}_g(M)^{\frac{n-1}{n}}$.

Let us return to Theorem 2.4. We may extract minimal hypersurfaces Σ from the Allen–Cahn ε-solutions $u_{\varepsilon,p}$ in it by fixing p, sending $\varepsilon \to 0$, and invoking the following powerful compactness theorem, which is the culmination of the works of Hutchinson–Tonegawa ([24]), Tonegawa–Wickramasekera ([25]), Guaraco ([19]), and Gaspar ([26]):

Theorem 2.6 ([24, 25, 19, 26]) *Fix (M^n, g), $3 \le n \le 7$. Let $\varepsilon_i \to 0$ and u_{ε_i} be Allen–Cahn ε_i-solutions satisfying:*

- $\sup_i E_{\varepsilon_i}[u_{\varepsilon_i}] < \infty$*, and*
- $\sup_i \operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}] < \infty$.

Then, there exist disjoint closed embedded minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_k \subset (M^n, g)$ *and integers* $m_1, \ldots, m_k \geq 1$ *so that, subsequentially:*

(a) $u_{\varepsilon_i} \rightsquigarrow m_1\Sigma_1 \cup \cdots \cup m_k\Sigma_k$ *as* $\varepsilon_i \to 0$*, and*
(b) $\operatorname{ind}[\Sigma_1] + \ldots + \operatorname{ind}[\Sigma_k] \leq \lim_i \operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}]$.

Above, the case $n \geq 8$ is omitted due to the usual presense of a codimension-8 singular set in minimal hypersurfaces, while the case $n = 2$ is omitted due to a serious subtlety that will be addressed later.

The integers $m_1, \ldots, m_k \geq 1$ are called the corresponding **multiplicities** of the components $\Sigma_1, \ldots, \Sigma_k$. Whether or not all multiplicities have to equal 1 plays a fundamental role in the applicability of these methods in counting problems for minimal hypersurfaces. See Section 4 for more.

It turns out that high-multiplicity solutions can, in general, occur in limits of Allen–Cahn ε-solutions. The following is a theorem due to del Pino–Kowalczyk–Wei–Yang ([27]):

Theorem 2.7 ([27]) *Fix a closed* (M^n, g) *and let* $\Sigma \subset (M^n, g)$ *be a closed embedded nondegenerate minimal hypersurface and* $m \geq 1$ *be an integer so that:*

- $|A_\Sigma|^2 + \operatorname{Ric}_g > 0$ *on* Σ*, and*
- Σ *is separating if* m *is odd.*

Then, there exist $\varepsilon_i \to 0$ *and Allen–Cahn* ε_i*-solutions* u_{ε_i} *so that:*

$$u_{\varepsilon_i} \rightsquigarrow m\Sigma \text{ as } \varepsilon_i \to 0.$$

It is worth remarking here that it was conjectured in [27] that the index of u_{ε_i}, i.e., $\operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}]$, diverges as $i \to \infty$.

3 Multiplicity-one theorem

Our main joint theorem with O. Chodosh is that high multiplicity will not occur in the 3-dimensional setting of Theorem 2.6, where index bounds *are* assumed, if the limiting $\Sigma_1, \ldots, \Sigma_k$ (or rather, their two-sided covers) are known to be nondegenerate:

Theorem 3.1 ([28]) *Assume the setting of Theorem 2.6, with* $n = 3$*. Then, we have in addition to Theorem 2.6 (a) and (b):*

(c) *Away from* $\leq \lim_i \operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}]$ *points on* Σ, $\{u_{\varepsilon_i} = 0\}$ *decomposes as disjoint graphs converging in* $C^{2,\alpha}_{\mathrm{loc}}$ *to* Σ *with suitable multiplicities,*
(d) *some* $m_i \geq 2 \implies$ *there exists a positive Jacobi field on two-sided cover* Σ_i' *of* Σ_i *(thus,* $\operatorname{ind}[\Sigma_i'] = 0$ *and* $\operatorname{nul}[\Sigma_i'] = 1$*), and*
(e) *all* $m_i = 1 \implies \operatorname{ind}[\Sigma] + \operatorname{nul}[\Sigma] \geq \lim_i \operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}] + \operatorname{nul}_{\varepsilon_i}[u_{\varepsilon_i}]$.[2]

[2]We note that this particular conclusion, (e), was proven in all dimensions.

In particular, high multiplicity does *not* occur when the metric g is "bumpy," i.e., when no immersed minimal surface carries nontrivial Jacobi fields. In particular, Theorem 3.1 resolved a strong form of a "multiplicity one" conjecture of Marques–Neves, in the setting of Allen–Cahn. Note that the "bumpy" condition is generic in the sense of Baire category, as shown by White in [29].

One main ingredient of the theorem, for conclusions (c) and (d), is based on sharpening some recent novel work of Wang–Wei [30] that resolved the "finite index implies finitely many ends" Allen–Cahn conjecture in $\mathbf{R}^2$. Other than generalize these to the 3-dimensional setting (albeit with energy bounds), we had to improve the obtained regularity to ensure that the level sets are close enough to being minimal to construct Jacobi fields.

Another main ingredient, for conclusion (e), was to study a refined expansion of the second variation operator to obtain an inequality that goes in the opposite direction relative to Theorem 2.6's (b), and which is sensitive to the fact that our multiplicity equals one. See also work of Alikakos–Fusco–Stefanopoulos ([31]) and del Pino–Kowalczyk–Wei ([14]), though here an added difficulty is that in our case the u_{ε_i} are essentially arbitrary and not constructed by us.

4 Geometric implication

Finally we point out an important consequence of this work to the study of minimal surfaces in closed Riemannian 3-manifolds (M^3, g). Let us assume that the metric g on M is bumpy, i.e., that $\mathrm{nul}[\Sigma] = 0$ for all closed immersed minimal Σ. Fixing p and applying Theorem 2.4 for sufficiently small ε and inputting the sequence of $u_{\varepsilon,p}$ with $\varepsilon \to 0$ into Theorems 2.6, 3.1, we find that the multiplicity-one conclusion does indeed hold, so there exists an embedded hypersurface Σ_p (possibly disconnected) so that:

$$\mathrm{Area}_g(\Sigma_p) \sim p^{1/3}, \ \mathrm{ind}[\Sigma_p] = p.$$

Noting that Σ_p may be disconnected, nonetheless there exists at least one connected component $\Sigma'_p \subset \Sigma_p$ with

$$\mathrm{Area}_g(\Sigma'_p) \lesssim p^{1/3}, \ \mathrm{ind}[\Sigma_p] \geq p^{2/3}. \tag{4.1}$$

Combining (4.1) and work of Ejiri–Micallef ([32]) yields

$$\mathrm{genus}(\Sigma'_p) \geq \tfrac{1}{6} p^{2/3} - O(p^{1/3}).$$

In particular, this resolved a special case of a conjecture due to Yau in the three-dimensional case and for generic metrics:

Conjecture 4.1 (Yau's conjecture) *Any closed (M^n, g), $3 \leq n \leq 7$, contains infinitely many distinct closed embedded minimal hypersurfaces.*

Irie–Marques–Neves had previously resolved, in [33], Conjecture 4.1 for generic metrics using the Liokumovich–Marques–Neves Weyl law, from [23], for the Almgren–Pitts width spectrum. An analogous Allen–Cahn strategy was subsequently taken by Gaspar–Guaraco once they obtained Theorem 2.5. Our proof of existence of infinitely many embedded surfaces also carries through when $\mathrm{Ric}_g > 0$ and $n = 3$; see also the previous Almgren–Pitts work of Marques–Neves ([34]) when $\mathrm{Ric}_g > 0$ and $3 \leq n \leq 7$. Song finally resolved Conjecture 4.1 in full using a novel modification of the Almgren–Pitts width spectrum in [35].

5 Above dimension three

Note that Theorem 3.1 leaves open the case $4 \leq n \leq 7$, even though it was handled by Theorem 2.6. There is no evidence to suggest that Theorem 3.1 will break in these dimensions. In fact, recent work of Wang–Wei ([36]) can be plugged into the proof of Theorem 3.1 in [28] and can extend it to $4 \leq n \leq 7$, if the following ingredient were provable:

Conjecture 5.1 (Allen–Cahn Stable Bernstein Conjecture) *If $4 \leq n \leq 7$ and $u : \mathbf{R}^n \to \mathbf{R}$ is an Allen–Cahn 1-solution, which is:*

- *stable, i.e., $\mathrm{ind}_1[u] = 0$ for compactly supported variations, and*
- *of Euclidean energy growth, i.e., $\limsup_{R\to\infty} R^{1-n} \int_{B_R} |\nabla u|^2 < \infty$,*[3]

then u is a function of one variable.

If, indeed, Conjecture 5.1 is true for $4 \leq n \leq 7$, then Theorem 3.1 automatically extends to the corresponding dimension together with the help of Wang–Wei's recent work in [36].

This conjecture was proven for $n = 2$ by Ghoussoub–Gui in [37] even without the Euclidean energy growth assumption. For $n = 3$, it was proven by Ambrosio–Cabré in [38]. For $n \geq 8$, this conjecture is known to not hold by work of Pacard–Wei in [39]; see also the work of Liu–Wang–Wei ([40]).

6 Dimension two

Note that Theorem 2.6 was not stated for dimension $n = 2$. This is because current techniques can only show that one has convergence $u_{\varepsilon_i} \rightsquigarrow \Sigma$, not to an immersed geodesic Σ, but rather to some **stationary geodesic network** Σ (again, possibly with multiplicity) with $\leq \lim_i \mathrm{ind}_{\varepsilon_i}[u_{\varepsilon_i}]$ vertices altogether. This remains an open question:

Question 6.1 *Does Theorem 2.6 (a) extend to $n = 2$, but with $\Sigma_1, \ldots, \Sigma_k$ not necessarily disjoint? What is the appropriate extension of (b)?*

The answer to (a) Question 6.1 is currently only known for general double-well potentials W, or even the special case $W(t) = \frac{1}{4}(1 - t^2)^2$, when $\sup_i \mathrm{ind}_{\varepsilon_i}[u_{\varepsilon_i}] \leq 1$ by work of the author in [41].

Recently, O. Chodosh and the author obtained an affirmative answer to (a) of Question 6.1 in [42] for a *special* choice of potential W coming from the theory

[3] This assumption can be dropped, if one seeks a stronger conjecture.

of integrable systems but which still satisfies all relevant double-well properties. We made diligent use of new results of Liu–Wei ([43]) for this particular integrable potential. Specifically, we proved:

Theorem 6.2 ([42]) *Fix a closed* (M^2, g). *Let* $\varepsilon_i \to 0$ *and* u_{ε_i} *be Allen–Cahn* ε_i*-solutions with respect to the integrable potential* $W(t) = 1 - \cos(\pi t)$, *and which satisfy*

- $\sup_i E_{\varepsilon_i}[u_{\varepsilon_i}] < \infty$, *and*
- $\sup_i \operatorname{ind}_{\varepsilon_i}[u_{\varepsilon_i}] < \infty$.

Then, there exist distinct simple closed geodesics $\sigma_1, \ldots, \sigma_k \subset (M^2, g)$ *and integers* $m_1, \ldots, m_k \geq 1$ *so that, subsequentially:*

$$u_{\varepsilon_i} \rightsquigarrow m_1\sigma_1 \cup \cdots \cup m_k\sigma_k \text{ as } \varepsilon_i \to 0.$$

In the same paper, we also compute the constant $\alpha(2) = \sqrt{\pi}$ in Theorem 2.5 (with $n = 2$). To do so, we invoke Theorem 6.2 on carefully chosen perturbations of a round two-sphere and import its conclusion into the Almgren—Pitts min-max theory using work of Dey ([44]) and invoke a novel Lyusternik–Schirelman-inspired counting argument. For more information, we refer the reader to [42].

It is worth noting that recent constructions of Liu–Pacard–Wei ([45]) show that the multiplicity-one result of Theorem 3.1 does not hold in the two-dimensional case. This makes the two-dimensional case all the more interesting.

7 Two parallel stories

The author is aware of some very interesting developments in two fields that are roughly parallel to the Allen–Cahn equation discussed here.

One is the study of critical points of the s-perimeter with $s \in (0, 1)$, also known as non-local minimal hypersurfaces, and their convergence to minimal hypersurfaces as $s \to 1$. Caselli–Florit Simon–Serra established in [46] an analog to Theorem 2.4. Recently, Florit Simon established in [47] the s-perimeter analog of Theorem 2.4 (d) and he used to to prove an analog of the multiplicity-one result from Theorem 3.1 for $n = 3$. Conjecture 5.1 is known in the s-perimeter setting up to $n = 4$; it was shown for $n = 3$ by Cabré–Cinti–Serra in [48] and for $n = 4$ by Chan–Dipierro–Serra–Valdinoci in [49]. One expects that the analog of the higher dimensional Wang–Wei estimates ([36]) to soon be available in the s-perimeter setting and allow an extension of the Florit Simon result up to $n = 4$.

The second is the study of ε-solutions of the self-dual magnetic Ginzburg–Landau equation (abelian Higgs model) and their possible convergence as $\varepsilon \to 0$ to codimension-two minimal submanifolds; cf. Definition 2.1. This theory is relatively underdeveloped but of significant interest. In recent progress, Pigati–Stern proved in [50] the analog to the Hutchinson–Tonegawa precursor to Theorem 2.6 from [24]: no index bounds are assumed, and the convergence is accordingly to a weaker object, a "codimension-two integral stationary varifold". It is not yet clear how to exploit index bounds in this equation. The analog to Conjecture 5.1 is also wide open when $n \geq 3$; if $n = 2$, then a full classification of all critical points (in the self-dual case) was obtained by Taubes in [51].

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